TECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 65 NUMBER 10

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DECEMBER 1958

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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THIS IS THE OFFICIAL JOURNAL OF THE

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For non-members the subscription price is $6.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y., during the months of January, February, March, April, May, June-July, August-September, October, November, December.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

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PRINTED IN THE UNITED STATES OF AMERICA

FACTORIAL $\frac{1}{2}$: A SIMPLE GRAPHICAL TREATMENT*

FREDERICK MOSTELLER AND D. E. RICHMOND,
Harvard University and Williams College

The great secret of successful interpolation or extrapolation is the choice of a good scale for plotting the function or the independent variable. Since the straight line is the only curve that provides easy and reliable interpolation, a good scale is one that tends to produce a straight line. We use this principle twice to obtain an approximate numerical definition for $\frac{1}{2}!$.

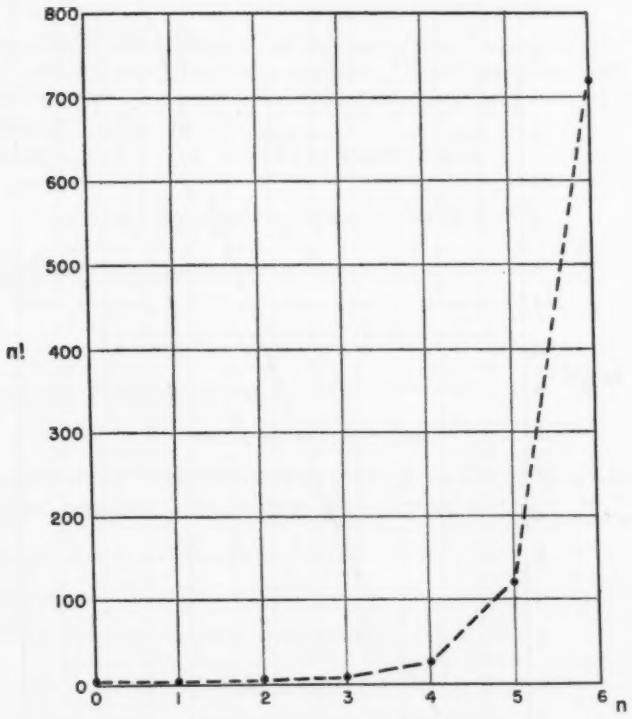

FIG. 1. $n!$ plotted against $n(=0, 1, 2, 3, 4, 5, 6)$.

Elementary students, after having heard about $n!$ for integers, often wish to know its value when n is not an integer. Our discussion is keyed to the definition and evaluation of $\frac{1}{2}!$, and more generally of $(n+\frac{1}{2})!$, where n is an integer. While the discussion could be extended beyond the half-integers, its merit lies in its

* This research was facilitated by a grant from the Ford Foundation and by the Laboratory of Social Relations of Harvard University.

intuitive appeal—to pursue this extension would be excessive.

We begin by plotting $f(n)=n!$ against n for a few values of n (see Fig. 1).

The request to define $(n+\frac{1}{2})!$ is equivalent to asking how to interpolate certain values on a curve connecting the points plotted in Figure 1 for integral values of n. One feels that a complete set of interpolated points should create a smooth curve, but words like "smooth" are vague, except when points are collinear. Then the natural step is to put in the extra points by linear interpolation.

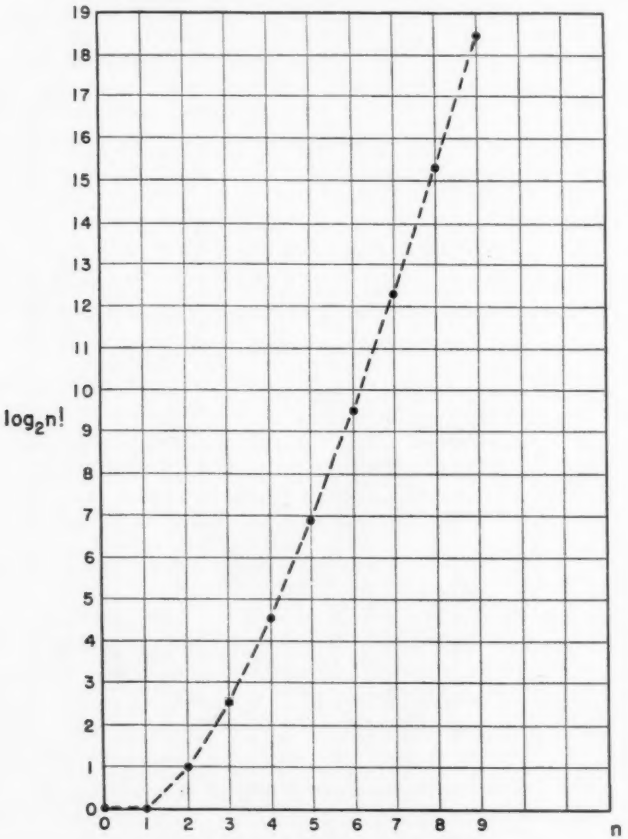

FIG. 2. Plot of $\log_2 n!$ against $n(=0, 1, \cdots, 9)$.

Our trouble is that the function $f(n)$ rises so fast that we cannot get a good "purchase" on it. If a function increases too fast for comfort, it is natural to take its logarithm, so we plot $\log_2 n!$ against n (see Fig. 2).

The curve passed by the eye through these new points seems to be getting straighter as n increases. We check this by looking at the slopes of the chords between n and $n+1$ and between $n+1$ and $n+2$; namely:

$$\log (n+2)! - \log (n+1)! = \log (n+2),$$
$$\log (n+1)! - \log n! = \log (n+1).$$

We observe that as n gets large, the difference in the slopes of adjacent chords

$$\log (n+2) - \log (n+1) = \log \left(1 + \frac{1}{n+1}\right)$$

becomes very small. Therefore, as n gets larger, the polygonal curve formed by the chords is getting straighter and straighter. This straightness means that we can evaluate $\log (n+\frac{1}{2})!$ by going far enough out on the curve, so that linear interpolation will do the trick.

So for large integral n we would have approximately

$$\log \left(n + \frac{1}{2}\right)! = \frac{1}{2} [\log n! + \log (n+1)!]$$

or, the equivalent approximation

$$\left(n + \frac{1}{2}\right)! = \sqrt{n!(n+1)!} = n!\sqrt{n+1}. \tag{1}$$

Since the fundamental relation

$$(n+1)! = (n+1)n! \tag{2}$$

serves to define $n!$ for integral values of $n \geqq 0$ (given $1!=1$), it is natural to try to retain it for nonintegral values of n. We therefore write

$$\frac{3}{2}! = \frac{3}{2} \cdot \frac{1}{2}!,$$
$$\frac{5}{2}! = \frac{5}{2} \cdot \frac{3}{2}! = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}!,$$
$$\cdots\cdots\cdots\cdots,$$
$$\left(n + \frac{1}{2}\right)! = \frac{(2n+1)(2n-1)\cdots 3}{2^n} \cdot \frac{1}{2}!. \tag{3}$$

By inserting the factors $2n, 2n-2, \cdots, 2$ in the numerator and denominator of the right-hand side of (3) we can rewrite it as

$$\left(n + \frac{1}{2}\right)! = \frac{(2n+1)!}{2^{2n} n!} \cdot \frac{1}{2}!.$$

Then we have

$$\frac{1}{2}! = \frac{2^{2n}n!}{(2n+1)!}\left(n+\frac{1}{2}\right)!. \tag{4}$$

We obtain a sequence of approximations $v_0, v_1, v_2, \cdots$ to the desired $\frac{1}{2}!$ by using (1) as an approximation for $(n+\frac{1}{2})!$ in (4). Then

$$v_n = \frac{2^{2n}(n!)^2\sqrt{n+1}}{(2n+1)!}.$$

It is pleasant to work entirely with integers. Accordingly, we shall compute v_n^2 instead of v_n:

$$v_n^2 = \frac{(n+1)(n!)^4 2^{4n}}{[(2n+1)!]^2}. \tag{5}$$

Each member of this sequence is positive. Since

$$\frac{v_n^2}{v_{n-1}^2} = 1 - \frac{1}{(2n+1)^2}, \qquad n \geqq 1, \tag{6}$$

the sequence decreases monotonically. Hence v_n^2 approaches a limit which will be defined as $(\frac{1}{2}!)^2$.

Equation (6) is equivalent to

$$v_{n-1}^2 = \left(1 + \frac{1}{4n(n+1)}\right)v_n^2, \tag{7}$$

which will be useful later.

We wish to evaluate this limit graphically. We need to adopt a scale on the horizontal axis so that as n goes from 1 to ∞, the graph stays on the paper. The classical device is to plot against $1/n$ or more generally $1/(n+k)$ (k a positive constant), because then as n goes from 1 to ∞, the function is plotted for points on the interval from $1/(1+k)$ to 0. If by good luck we can choose k so that the resulting graph is approximated by a straight line, we can find the required limit of v_n^2 as $n \to \infty$, by finding where this straight line crosses the vertical axis.

The first three points $(1/(1+k), v_1^2)$, $(1/(2+k), v_2^2)$, $(1/(3+k), v_3^2)$ can be lined up exactly by choosing $k = 23/25 = 0.92$. This value of k is near enough to unity to suggest plotting v_n^2 against $1/(n+1)$ (see Fig. 3). (Had $n = 0, 1, 2$ been used for the first three points rather than $n = 1$, 2, and 3, k would have been 16/17 or approximately 0.94, even closer to unity. By not plotting the point (1, 1) corresponding to $n = 0$ in Figure 3, we have been able to enlarge the scale on both axes.)

The result is a satisfactory resemblance to a straight line, especially on the left. Extrapolating to $1/(n+1) = 0$ by eye gives the approximation 0.785 for $[(\frac{1}{2})!]^2$. The straightness is sufficiently encouraging to justify a more careful extrapolation to 0.

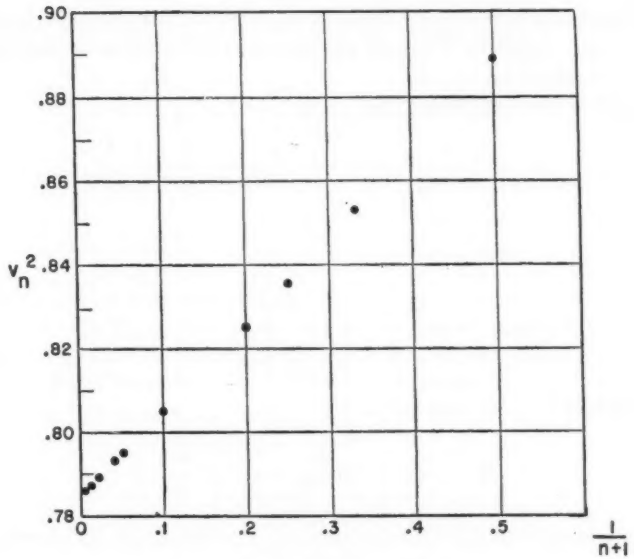

FIG. 3. Plot of v_n^2 against $1/(n+1)$.

TABLE 1

ESTIMATES OF $(\frac{1}{2}!)^2=v_\infty^2$

n	$\frac{1}{n+1}$	First approximation	Second approximation
		v_n^2	$b_n=v_n^2\left(1-\frac{1}{4(n+1)}\right)$
1	.5	.88888 88889	.77777 77778
2	.33 · · ·	.85333 33333	.78222 22222
3	.25	.83591 83673	.78367 34694
4	.20	.82559 83875	.78431 84681
9	.10	.80527 22492	.78514 04430
19	.05	.79527 62214	.78533 52686
24	.04	.79329 10176	.78535 81074
49	.02	.78933 49223	.78538 82477
99	.01	.78736 41070	.78539 56968
199	.0050	.78638 05239	.78539 75483
399	.0025	.78588 91906	.78539 80098

$$\frac{\pi}{4} = .78539\ 81634$$

To facilitate the work, we tabulate v_n^2 against $1/(n+1)$ for certain convenient values of n (see Table 1). The last column contains extrapolations to v_∞^2 which will now be explained.

The slope of the line joining the point

$$P_n = \left(\frac{1}{n+1}, v_n^2\right) \quad \text{to} \quad P_{n-1} = \left(\frac{1}{n}, v_{n-1}^2\right)$$

is

$$m_n = \frac{v_{n-1}^2 - v_n^2}{\dfrac{1}{n} - \dfrac{1}{n+1}}.$$

Using equation (7),

$$m_n = n(n+1)[v_{n-1}^2 - v_n^2] = \frac{v_n^2}{4},$$

a remarkably simple result. The line through P_n with slope m_n intersects the v_n^2 axis at

$$b_n = v_n^2\left[1 - \frac{1}{4(n+1)}\right].$$

The results for b_n have been listed in Table 1 as estimates of $v_\infty^2 = (\frac{1}{2}!)^2$. As we go down the table we note that the value is stabilizing near 0.785398, which is $\pi/4$ to six decimal places. The correct result $(\frac{1}{2})! = \sqrt{\pi}/2$ is therefore suggested. This result is no accident, since there is, in fact, a close connection between our work and Wallis' formula

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$$

which may be written [2]

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{(2n!)^2(2n+1)}$$

$$= \lim_{n\to\infty} \left(\frac{2n+1}{n+1}\right)(v_n^2),$$

so that $\lim_{n\to\infty} v_n^2 = \pi/4$. The interest in this note lies in the elementary methods which have been used to evaluate $(\frac{1}{2})!$, rather than in the result itself.

The remainder of this note is concerned with improvements of the results in Table 1 which may be obtained by plotting against $1/(n+k)$ where $k \neq 1$.

If we parallel the previous discussion, we obtain for the slope of the line from $P_n=(1/(n+k), v_n^2)$ to $P_{n-1}=(1/(n+k-1), v_{n-1}^2)$, the value

$$m_n = \frac{v_n^2}{4}\,\frac{(n+k-1)(n+k)}{n(n+1)}, \tag{8}$$

and for the intercept on the v_n^2 axis

$$b_n = v_n^2\left[1 - \frac{1}{4(n+1)} + \frac{1-k}{4(n+1)n}\right]. \tag{9}$$

What is the best k to use for a given n? As long as the graph is concave upward to the left of P_n, b_n will be too small. Since b_n increases as k decreases, we shall obtain the best value of b_n among graphs of this type by choosing the smallest k compatible with the requirement of upward concavity.

The graph is concave upward to the immediate left of P_n if the ratio $R=m_n/m_{n+1}\geqq 1$, that is, using (8), if

$$R = \frac{\left(n+\frac{3}{2}\right)^2(n+k-1)}{n(n+1)(n+k+1)} \geqq 1,$$

TABLE 2

BOUNDS FOR $(\frac{1}{2}!)^2=v_\infty^2$ OBTAINED FROM THE BEST VALUES OF k

$$b_n=v_n^2\left(1-\frac{1}{4(n+1)}+\frac{1-k}{4(n+1)n}\right)$$

	Lower bounds	Upper bounds
n	$k=\frac{7n+9}{8n+9}$	$k=\frac{7}{8}$
1	.78431 37255	.79166 66667
2	.78506 66667	.78666 66667
3	.78525 66481	.78585 03401
4	.78532 52954	.78560 84656
9	.78538 89838	.78542 00514
19	.78539 70137	.78540 06696
24	.78539 75746	.78539 94246
49	.78539 80898	.78539 83157
99	.78539 81542	.78539 81821
199	.78539 81622	.78539 81657
399	.78539 816325	.78539 81637

$$\frac{\pi}{4} = .78539\ 816340$$

which implies that $k \geqq (7n+9)/(8n+9)$. Note that if this condition is satisfied for a given n, it is satisfied for all larger values of n and hence for the whole graph to the left of P_n. To obtain the best value of v_n^2 for a fixed n among graphs with upward concavity, we should therefore use $k_n = (7n+9)/(8n+9)$. Results for some values of n are given in Table 2.

As we know, even this most favorable k gives too small a result for v_n^2. It is easy to obtain a result which is known to be too *large* by choosing $k \leqq 7/8$, since for all such values of k the graph is concave *downward* to the left of P_n. Among these values, $k = 7/8$ gives the smallest b_n and hence the best upper bound for v_n^2 among graphs of this type. Values of b_n for $k = 7/8$ have been added to Table 2.

Unfortunately, a choice of k *between* $7/8$ and k_n corresponds to a graph which changes its concavity to the left of P_n so that no further improvement can be made by these elementary methods since we have no way to determine whether the result is too large or too small. Nevertheless, the bounds obtained are rather good.

In both tables the entries are believed to be accurate to the number of places indicated. Tables of logarithms and logarithms of factorials used in these calculations were [1, 3]. The authors wish to express their appreciation to Mrs. Cleo Youtz for carrying out the calculations.

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SINGULAR VALUES OF A MATRIX

ALI R. AMIR-MOÉZ, Queens College, AND ALFRED HORN, University of California, Los Angeles

1. The singular values. There is an analogy between matrices and complex numbers which has been known for a long time. In this analogy, the operation which leads from a matrix A to its adjoint A^* corresponds to passing to the conjugate for complex numbers. Accordingly, Hermitian matrices (those for which $A = A^*$) play the roles of real numbers. Positive semidefinite matrices correspond to nonnegative real numbers, and we shall refer to such matrices as nonnegative matrices. Every matrix A can be written uniquely in the form $H_1 + iH_2$, where H_1 and H_2 are Hermitian. Indeed, $H_1 = (A + A^*)/2$, $H_2 = (A - A^*)/2i$. Unitary matrices correspond to complex numbers of modulus one. Moreover, every unitary matrix can be written in the form e^{iH}, where H is Hermitian and the

exponential is defined by its power series.

There is also an analogue to the polar form of a complex number: every matrix A may be written as $A = UH_1 = H_2U$, where U is unitary, and H_1, H_2 are nonnegative Hermitian. H_1 and H_2 are uniquely determined as the nonnegative Hermitian matrices whose squares are A^*A, AA^* respectively. The unitary part is uniquely defined only when A is nonsingular.

The analogy is deepened when we consider the eigenvalues of the matrices. A Hermitian matrix has only real eigenvalues, and the eigenvalues are all nonnegative if and only if the matrix is nonnegative. The eigenvalues of a unitary matrix have modulus one.

In this paper, we are going to consider the following aspect of the analogy. Since much more is known about the eigenvalues of a Hermitian matrix, it is natural to ask to what extent the eigenvalues of the Hermitian matrices related to A in the above decompositions determine the eigenvalues of A. Let us define the eigenvalues in question.

DEFINITION. *The absolute singular values of A are the eigenvalues of the nonnegative part of the polar decomposition of A. Equivalently they are the nonnegative square roots of the eigenvalues of A^*A.*

The eigenvalues of AA^* are the same as those of A^*A, so that it does not matter which polar decomposition of A we use.

There are two norms for a matrix which are in common use. It is noteworthy that both may be expressed in terms of the absolute singular values ρ_i, where $\rho_1 \geqq \cdots \geqq \rho_n$. The Hilbert norm is defined by

$$\|A\|_H^2 = \sup_{|x|=1} (Ax, Ax),$$

and therefore

$$\|A\|_H^2 = \sup_{|x|=1} (A^*Ax, x) = \rho_1^2.$$

The Frobenius norm is

$$\|A\|_F^2 = \sum_{i,j=1}^{n} |a_{ij}|^2,$$

where a_{ij} is the i, j element of A. Since the ith diagonal element of A^*A is $\sum_{j=1}^{n} |a_{ij}|^2$, we see that

$$\|A\|_F^2 = \text{trace } A^*A = \sum_{i=1}^{n} \rho_i^2.$$

More generally, by the well-known theorem of Schur [7] on the connection between the diagonal elements and the eigenvalues of a Hermitian matrix, we have

$$\rho_n^2 + \cdots + \rho_{n-k+1}^2 \leqq \sum_{p=1}^{k} \sum_{j=1}^{n} |a_{i_p,j}|^2 \leqq \rho_1^2 + \cdots + \rho_k^2, \qquad 1 < k < n,$$

where $i_1, \cdots, i_k$ is any increasing sequence of integers between 1 and n.

DEFINITION. *The real singular values of A are the eigenvalues of $(A+A^*)/2$, and the imaginary singular values of A are the eigenvalues of $(A-A^*)/2i$.*

If A is a normal matrix, then it is unitarily equivalent to a diagonal matrix and consequently it is easy to see that the real, imaginary and absolute singular values of A are merely the real parts, imaginary parts and absolute values of the eigenvalues of A. In the general case the relations between these quantities are given in the following theorems.

THEOREM 1. *Let A be a matrix with eigenvalues λ_i such that $\Re\lambda_1 \geqq \cdots \geqq \Re\lambda_n$ and real singular values $\alpha_1 \geqq \cdots \geqq \alpha_n$. Then*

$$(1) \qquad \Re\lambda_1 + \cdots + \Re\lambda_k \leqq \alpha_1 + \cdots + \alpha_k, \qquad 1 \leqq k \leqq n,$$

$$(2) \qquad \Re\lambda_1 + \cdots + \Re\lambda_n = \alpha_1 + \cdots + \alpha_n.$$

Conversely, if λ_i are any complex numbers and $\alpha_1 \geqq \cdots \geqq \alpha_n$ are real numbers satisfying (1) *and* (2), *there exists a matrix A with eigenvalues λ_i and real singular values α_i.*

Proof. (The first part of this theorem is due to Ky Fan [2].) If A has eigenvalues λ_i and real singular values α_i, then there exists a triangular matrix

$$T = \begin{pmatrix} \lambda_1 & & & \\ & \cdot & 0 & \\ & * & \cdot & \\ & & & \lambda_n \end{pmatrix}$$

with zeros above the diagonal such that $A = U^*TU$, U unitary. Therefore

$$\frac{A + A^*}{2} = U^*\left(\frac{T + T^*}{2}\right)U.$$

But $(T+T^*)/2$ is Hermitian and has diagonal elements $\Re\lambda_i$ and eigenvalues α_i. Now in [5], Theorem 5, it was proved that (1) and (2) hold if and only if there exists a Hermitian matrix with diagonal elements $\Re\lambda_i$ and eigenvalues α_i. Thus (1) and (2) hold.

Conversely if (1) and (2) hold, let $H=(h_{ij})$ be a Hermitian matrix with eigenvalues α_i and diagonal elements $\Re\lambda_i$. Let

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 2h_{21} & \lambda_2\ 0 & \cdots & 0 \\ \cdots & & \cdots & \\ 2h_{n1} & \cdots & 2h_{n-1,n} & \lambda_n \end{pmatrix}.$$

Then A has eigenvalues λ_i and $(A+A^*)/2=H$. This completes the proof.

THEOREM 2. *A necessary and sufficient condition that there exist a matrix A with imaginary singular values $\beta_1 \geqq \cdots \geqq \beta_n$ and eigenvalues λ_i such that $\Im\lambda_1 \geqq \cdots \geqq \Im\lambda_n$ is*

$$\Im\lambda_1 + \cdots + \Im\lambda_k \leqq \beta_1 + \cdots + \beta_k, \qquad 1 \leqq k < n, \tag{3}$$

$$\Im\lambda_1 + \cdots + \Im\lambda_n = \beta_1 + \cdots + \beta_n. \tag{4}$$

Proof. The proof is similar to that of Theorem 1.

THEOREM 3. *A necessary and sufficient condition that there exist a matrix A with absolute singular values $\rho_1 \geqq \cdots \geqq \rho_n$ and eigenvalues λ_i such that $|\lambda_1| \geqq \cdots \geqq |\lambda_n|$ is*

$$|\lambda_1 \cdot \cdots \cdot \lambda_k| \leqq \rho_1 \cdot \cdots \cdot \rho_k, \qquad 1 \leqq k \leqq n, \tag{5}$$

$$|\lambda_1 \cdot \cdots \cdot \lambda_n| = \rho_1 \cdot \cdots \cdot \rho_n. \tag{6}$$

Proof. The necessity was proved by Weyl [8], and the sufficiency is due to Horn [4].

We turn now to the question of the relationship between the eigenvalues of A and the eigenvalues of the unitary part of the polar decomposition of A. Conditions (5) and (6) are equivalent to the following condition: For each $k \leqq n$, the absolute value of the product of any k eigenvalues of A (with different subscripts) lies in the convex set generated by the products k at a time of the absolute singular values. We have a corresponding result in the next theorem.

THEOREM 4. *Let A be a nonsingular matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$, and let $\gamma_1, \cdots, \gamma_n$ be the eigenvalues of the unitary part U of the polar decomposition of A. Then for each $k \leqq n$, the product of any k of the λ_i lies in the convex cone generated by the products k at a time of the γ_i.*

Proof. A convex cone (with vertex at the origin) in the complex plane is a set of numbers which is closed under linear combinations with nonnegative coefficients. Let $A=UH$ be a polar decomposition of A. Suppose $Ax=\lambda x$, $x \neq 0$. Then $\lambda(x, Hx)=(Ax, Hx)=(UHx, Hx)=\sum_{i=1}^{n} \gamma_i |(Hx, u_i)|^2$, where $\{u_i\}$ is the orthonormal sequence of eigenvectors of U. Since A is nonsingular, so is H, and therefore $(x, Hx)>0$. Therefore λ lies in the convex cone generated by the γ_i. This completes the proof for $k=1$.

If $k>1$, let $A_{(k)}$ be the kth adjugate (exterior product) [6] of A. Then $A_{(k)}=U_{(k)}H_{(k)}$, and $U_{(k)}$ is unitary and $H_{(k)}$ is nonnegative. Furthermore the eigenvalues of $A_{(k)}$ are the products k at a time of the eigenvalues of A, and similarly for $U_{(k)}$. Therefore the statement for $k=1$ implies the statement for all $k \leqq n$. The proof is now complete.

We now consider the question of the converse of Theorem 4. For $n=2$, it is indeed true that if λ_1, λ_2 are nonzero complex numbers and γ_1, γ_2 are complex

numbers of modulus one such that

$$\frac{\lambda_1\lambda_2}{|\lambda_1|\,|\lambda_2|} = \gamma_1\gamma_2$$

and the λ_i lie in the convex cone generated by γ_1, γ_2, then there exists a matrix A with eigenvalues λ_1, λ_2 whose unitary part has eigenvalues γ_1, γ_2. However for $k>2$, the conditions of Theorem 4 are not sufficient. For example, if the eigenvalues of the unitary part of A are all real, then the number of positive eigenvalues of A must be the same as the number of positive eigenvalues of its unitary part. Even if we add this condition, we still do not have sufficient conditions.

It is to be remarked, though, that a knowledge of the eigenvalues of the unitary part of A imposes conditions only on the arguments of the eigenvalues of A. That is, if there exists a matrix A with eigenvalues λ_i, whose unitary part has eigenvalues γ_i, and if ρ_i are any positive numbers, then there exists a matrix B with eigenvalues $\rho_i\lambda_i$ whose unitary part has eigenvalues γ_i. This remark as well as the above comment on the lack of validity of the converse of Theorem 3 is due to Professor R. Steinberg, University of California, Los Angeles.

If we assume known both the real and imaginary singular values of A, then the eigenvalues of A will be subject to much stronger conditions than those implied by Theorems 1 and 2. Exactly how much can be said is still unknown. Wielandt [9] has determined the exact region in which an eigenvalue of A can lie if we are given the real and imaginary singular values. But the more difficult problem of determining the exact range of variation of the sequence of all the eigenvalues of A is not yet solved.

A theorem relating all three types of singular values is the following, which corresponds to the Pythagorean theorem.

THEOREM 5. *If A has real, imaginary and absolute singular values α_i, β_i and γ_i respectively, then*

$$\sum_{i=1}^{n} \alpha_i^2 + \sum_{i=1}^{n} \beta_i^2 = \sum_{i=1}^{n} \gamma_i^2.$$

Proof. We have

$$\left(\frac{A + A^*}{2}\right)^2 + \left(\frac{A - A^*}{2i}\right)^2 = \left(\frac{AA^* + A^*A}{2}\right).$$

The result now follows from the following facts:

a) The eigenvalues of the square of a matrix are the squares of the eigenvalues of the matrix.

b) The trace of a matrix (sum of its diagonal elements) is equal to the sum of its eigenvalues.

c) The trace of a sum of matrices is the sum of their traces.

2. Singular values of sums and products of matrices. In this section we consider the singular values of a sum or product of matrices. If A and B are matrices, then $(A+B)^*=A^*+B^*$. Therefore the question about the relation between the real or imaginary singular values of $A+B$ and those of A and B reduces to the corresponding question about the eigenvalues of a sum of Hermitian matrices. Much has been learned about this latter problem in recent years. The most general result published so far is the following [1].

THEOREM 6. *Let A and B be Hermitian matrices with eigenvalues $\alpha_1 \geqq \cdots \geqq \alpha_n$ and $\beta_1 \geqq \cdots \geqq \beta_n$, respectively. Let the eigenvalues of $A+B$ be $\gamma_1 \geqq \cdots \geqq \gamma_n$. Let $i_1 \leqq \cdots \leqq i_k$ and $j_1 \leqq \cdots \leqq j_k$ be any sequences of integers between 1 and n such that $i_p+j_p \geqq n+p$ for $p=1, \cdots, k$, where $k \leqq n$. Then*

$$(7) \quad \alpha'_{i_1} + \cdots + \alpha'_{i_k} + \beta'_{j_1} + \cdots + \beta'_{j_k} \leqq \gamma_{(i_1+j_1-n)'} + \cdots + \gamma_{(i_k+j_k-n)'},$$

where $i'_k=i_k$ and $i'_p=\min(i_p, i'_{p+1}-1)$ for $p=k-1, \cdots, 1$, and the sequences $\{j'_p\}$, $\{(i_p+j_p-n)'\}$ are similarly defined.

Unpublished work of A. Horn and A. Hoffman shows that there exist still more inequalities between the eigenvalues of $A+B$ and those of A and B. These inequalities are also linear. However the exact range of variation of the eigenvalues of $A+B$ has not yet been completely determined.

Wielandt has remarked that for any matrix A, the eigenvalues of the $2n$-rowed Hermitian matrix

$$\tilde{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

are $\pm\rho_1, \cdots, \pm\rho_n$, where the ρ_i are the absolute singular values of A. Since the operation $A \to \tilde{A}$ is linear, we can apply Theorem 6 to obtain inequalities between the absolute singular values of $A+B$ and those of A and B.

For products of matrices, there is a theorem very much like Theorem 6. If in Theorem 6, we let the α_i, β_i, γ_i be the absolute singular values of A, B and AB respectively, and we replace the sums in (7) by products, then we obtain a theorem due to Amir-Moéz [1].

There remains the question of the relationship between the eigenvalues of the unitary part of AB and the eigenvalues of the unitary parts of A and of B. One might expect the former to lie in a convex cone determined by products of the latter. But this conjecture is refuted by the following examples.

(1) Any rotation in the plane is a product of two reflections. Thus we can have two unitary matrices with eigenvalues ± 1 whose product has eigenvalues $e^{\pm i\theta}$, where θ is any real number.

(2) Let H and K be positive definite matrices which do not commute. The unitary parts of H and K are the identity matrix whose eigenvalues are 1. But if the eigenvalues of the unitary part U of HK were all 1, then U would be the identity matrix and HK would then be positive definite. This contradicts the

hypothesis that $HK \neq KH$, that is, that HK is not Hermitian.

In case U and V are unitary matrices and O does not lie in the convex hull of the eigenvalues of U, then one can prove something about the eigenvalues of UV. In fact if $UVx = \lambda x$, and $|x| = 1$, then $\lambda(x, Ux) = (UVx, Ux) = (Vx, x)$ and therefore $\lambda = (Vx, x)/(x, Ux)$. Let the eigenvalues of U and V be denoted by α_i and β_i respectively. Then (Ux, x) lies in the convex hull of the α_i, and is by hypothesis different from O. It follows that $1/(x, Ux)$ lies in the convex cone generated by the α_i. Thus every eigenvalue of UV is a product of a number in the convex cone generated by the α_i and a number in the convex cone generated by the β_i.

A. Horn and R. Steinberg have settled the question raised above concerning the eigenvalues of the unitary part of a matrix:

THEOREM. *If* $(\lambda_1, \cdots, \lambda_n)$ *and* $(\alpha_1, \cdots, \alpha_n)$ *are n-tuples of complex numbers with* $|\lambda_i| \neq 0$, $|\alpha_i| = 1$, *then there exists a matrix A with eigenvalues* λ_i *such that the unitary part of A has eigenvalues* α_i *if and only if*

$$\prod_{i=1}^{n} \frac{\lambda_i}{|\lambda_i|} = \prod_{i=1}^{n} \alpha_i$$

and either

(a) α_i *lie in a closed half plane with* 0 *on its boundary but not on any line through* 0 *and** $(\arg \lambda_1, \cdots, \arg \lambda_n) \prec (\arg \alpha_1, \cdots, \arg \alpha_n)$, *where we use a branch of the argument which is continuous in the closed half plane, or*

(b) α_i *lie on a line through* 0 *and* $\lambda_i/|\lambda_i|$ *are a rearrangement of the* α_i, *or*

(c) α_i *do not satisfy* (a) *or* (b).

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MODELS OF PROJECTIVE AND EUCLIDEAN SPACE

DAVID GANS, New York University

Although the number of possible models of Euclidean 3-space is without end, and likewise for models of projective 3-space, one finds in the literature no single model which puts both spaces completely before our eyes at the same time, exhibiting their topological relations in a simple and easy-to-visualize manner. Apart from its intrinsic mathematical interest and value, such a miniature would go far toward meeting a sorely-felt need in geometrical education. The customary method of trying to get college students to see the relation between the two spaces, for example, is to have them visualize Euclidean space in terms of the unbounded Cartesian model, and then to add ideal points to this model. While there is nothing wrong with this procedure from an abstract, logical viewpoint, it fails utterly to give the student a visual grasp of projective space. For he is expected to believe, among other things, that the ideal point on a straight line is a definite point which is reached by going infinitely far in either direction along the unbounded line! Surely the Cartesian model is worthless as a pictorial basis for beliefs such as this.

The present article shows that the kind of simultaneous model referred to above can be obtained quite simply by the use of a sphere. In the model which is described, the points, planes, and straight lines of Euclidean 3-space are represented, respectively, by the points interior to the sphere, certain half ellipsoids composed of these points, and certain half ellipses on these ellipsoids. On adding to this model the points on the surface of the sphere, diametrically opposite points being identified, we obtain a model of projective 3-space in which the closed half ellipsoids and closed half ellipses resulting from the specified addition of points represent projective planes and lines, respectively. These three-dimensional models are generalizations of two-dimensional ones, possessing similar advantages, which have already been described in this MONTHLY [2].

1. A transformation T of Euclidean 3-space. Consider the transformation T,

$$(1) \qquad x' = \frac{x}{\sqrt{x^2+y^2+z^2+1}}, \quad y' = \frac{y}{\sqrt{x^2+y^2+z^2+1}},$$

$$z' = \frac{z}{\sqrt{x^2+y^2+z^2+1}},$$

for arbitrary real values of x, y, z. Then

$$(2) \qquad x'^2 + y'^2 + z'^2 = \frac{x^2+y^2+z^2}{x^2+y^2+z^2+1} < 1.$$

Hence, T sends each point (x, y, z) into a point (x', y', z') within the unit sphere S with center at the origin O. From (2) we obtain

$$x^2 + y^2 + z^2 + 1 = \frac{1}{1 - x'^2 - y'^2 - z'^2},$$

and substitution of this in (1) gives the inverse of T:

$$x = \frac{x'}{\sqrt{1 - x'^2 - y'^2 - z'^2}}, \quad y = \frac{y'}{\sqrt{1 - x'^2 - y'^2 - z'^2}}, \tag{3}$$

$$z = \frac{z'}{\sqrt{1 - x'^2 - y'^2 - z'^2}}.$$

We see that to each point (x', y', z') interior to S there corresponds a point (x, y, z). Being continuous, together with its inverse, T is thus a topological transformation of Euclidean 3-space into the interior of S.

If P is any point at distance $r>0$ from O, it follows from (1) and (2) that the point $P' = T(P)$ is between O and P, and at the distance $r/\sqrt{r^2+1}$ from O. Hence, straight lines through O go into the parts of themselves interior to S, and spheres with center O into smaller spheres with center O. In other words, the compression of space into the interior of S effected by T takes place radially toward O, the amount of contraction depending only on r.

It follows from the foregoing properties that the orthogonality of two straight lines, one of which goes through O, is invariant under T. Except for this special situation, however, T generally distorts angles. Nor is the collinearity of points preserved except for points on straight lines through O. These things will be apparent from the discussion to follow, in which the effect of T on planes and the general straight line are considered.

2. The effect of T on planes. A plane through O,

$$ax + by + cz = 0, \tag{4}$$

is transformed by T into that much of the locus

$$(ax' + by' + cz')/\sqrt{1 - x'^2 - y'^2 - z'^2} = 0, \quad \text{or} \quad ax' + by' + cz' = 0,$$

as lies interior to S. Thus, (4) goes into a part of itself, namely, into the interior of a circular disc with diameter 2 and center at O. Conversely, to each such disc there corresponds a plane through O which is transformed into it.

As a first example of a plane not through O let us consider $x = d$, where $d \neq 0$. T sends this into

$$d = x'/\sqrt{1 - x'^2 - y'^2 - z'^2}, \tag{5}$$

which is that part of the ellipsoid

$$(1 + d^2)\frac{x'^2}{d^2} + y'^2 + z'^2 = 1 \tag{6}$$

for which x' has the same sign as d. Being symmetrical to the coordinate planes,

and having semimajor, semiminor, and semimean axes of lengths 1, 1, $d/\sqrt{1+d^2}$, respectively, this ellipsoid is interior to S, except where it meets S along the great circle in the yz-plane, and is divided into symmetrical hemiellipsoids by the circle. The locus of (5) is that one of these hemiellipsoids (excluding the circle) which lies on the same side of the plane of the circle as does the plane $x=d$. (The other hemiellipsoid, also exclusive of the circle, is the transform of the plane $x=-d$ by T.)

We next show that every other plane not through O is likewise transformed into a hemiellipsoid, moreover, one related to it in size, shape, and position exactly as the hemiellipsoid (5) is related to the plane $x=d$. Hence, consider the plane

$$ax + by + cz = d, \tag{7}$$

where $d\neq 0$ and $a^2+b^2+c^2=1$. Substitution in (3) shows that this plane is transformed into

$$\frac{ax' + by' + cz'}{\sqrt{1 - x'^2 - y'^2 - z'^2}} = d, \tag{8}$$

where, of course, $x'^2+y'^2+z'^2<1$. To determine the nature of this locus it is useful to consider the related locus

$$ax' + by' + cz' = d\sqrt{1 - x'^2 - y'^2 - z'^2} \tag{9}$$

for $x'^2+y'^2+z'^2\leqq 1$. The latter goes through the intersection of $ax'+by'+cz'=0$ and $\sqrt{1-x'^2-y'^2-z'^2}=0$, *i.e.*, of a plane through O and the sphere S. Thus (9) meets S in a great circle. The locus of (8) results from excluding this circle from (9).

Now, (9) is part of the locus of

$$(ax' + by' + cz')^2 = d^2(1 - x'^2 - y'^2 - z'^2), \tag{10}$$

or

$$(a^2 + d^2)x'^2 + (b^2 + d^2)y'^2 + (c^2 + d^2)z'^2 + 2abx'y' + 2acx'z' + 2bcy'z' = d^2. \tag{11}$$

To determine the nature of this locus we shall compare (11) with the general equation

$$Ax^2 + By^2 + Cz^2 + 2Hxy + 2Fxz + 2Gyz + 2Lx + 2My + 2Nz + K = 0. \tag{12}$$

This equation represents an ellipsoid if

$$D \equiv \begin{vmatrix} A & H & F \\ H & B & G \\ F & G & C \end{vmatrix} \neq 0, \qquad \Delta \equiv \begin{vmatrix} A & H & F & L \\ H & B & G & M \\ F & G & C & N \\ L & M & N & K \end{vmatrix} \neq 0,$$

and if the roots of the characteristic equation

$$r^3 - Ir^2 + Jr - D = 0 \tag{13}$$

have the same sign, where $I=A+B+C$ and $J=AB+BC+CA-F^2-G^2-H^2$ (see [1], p. 273).

Applying this to equation (11) we see that

$$A = a^2 + d^2, \quad B = b^2 + d^2, \quad C = c^2 + d^2,$$
$$H = ab, \quad F = ac, \quad G = bc, \quad K = -d^2, \quad L = M = N = 0,$$

from which we find, after substitution and simplification, that

$$D = d^2(a^2 + b^2 + c^2 + d^2), \qquad \Delta = -d^2D,$$
$$I = a^2 + b^2 + c^2 + 3d^2, \qquad J = d^2(2a^2 + 2b^2 + 2c^2 + 3d^2).$$

Since, by hypothesis, $d\neq 0$ and $a^2+b^2+c^2=1$, we see that $D\neq 0$, and hence that $\Delta\neq 0$. The coefficients of equation (13) being real, so are the roots (see [1], p. 259). None of these roots is zero since the constant term $-D\neq 0$, and none is negative since I, J, D are all positive. Equation (13) thus has three positive roots. The locus of (11) is therefore an ellipsoid.

We next show that this ellipsoid is symmetrical to the plane (4). Two points (x_1, y_1, z_1), (x_2, y_2, z_2) are symmetrical to this plane if and only if the four equations

$$x_1 - x_2 = ka, \quad y_1 - y_2 = kb, \quad z_1 - z_2 = kc,$$
$$ax_1 + by_1 + cz_1 = -(ax_2 + by_2 + cz_2)$$

are satisfied. If (x_1, y_1, z_1) is given, these equations determine x_2, y_2, z_2, k uniquely, the value of the last being

$$k = -2(ax_2 + by_2 + cz_2). \tag{14}$$

Hence, to prove the symmetry of the ellipsoid we can assume (x_1, y_1, z_1) is a point on (10), replace x_1, y_1, z_1 in (10) by x_2+ka, y_2+kb, z_2+kc, where k has the value in (14), and show that what results is (10) with x_2, y_2, z_2 replacing x', y', z'. The first of these steps gives

$$(ax_1 + by_1 + cz_1)^2 = d^2(1 - x_1^2 - y_1^2 - z_1^2),$$

and the second,

$$[a(x_2 + ka) + b(y_2 + kb) + c(z_2 + kc)]^2$$
$$= d^2[1 - (x_2 + ka)^2 - (y_2 + kb)^2 - (z_2 + kc)^2],$$

which reduces to

$$(ax_2 + by_2 + cz_2 + k)^2 = d^2[(1 - x_2^2 - y_2^2 - z_2^2) - 2k(ax_2 + by_2 + cz_2) - k^2].$$

On substituting from (14) into this we obtain

$$(ax_2 + by_2 + cz_2)^2 = d^2(1 - x_2^2 - y_2^2 - z_2^2),$$

which shows that (x_2, y_2, z_2) is on the ellipsoid (10). Hence the latter is symmetrical to the plane (4).

Returning to (9) we note that those of its points not on this plane lie on one side of it. For, any such point (x', y', z') of (9) gives to $ax'+by'+cz'$, which is the directed distance from the plane to this point, the value $d\sqrt{1-x'^2-y'^2-z'^2}$, which has the sign of d. This side of the plane (4) is also that on which the plane (7) lies, for any point (x', y', z') on the latter gives to $ax'+by'+cz'$ the value d. Now, the locus (9) is part of the ellipsoid (10), the remaining part is $ax'+by'+cz' = -d\sqrt{1-x'^2-y'^2-z'^2}$, and the latter clearly lies on the opposite side of (4). Hence the locus of (9) is that one of the two symmetrical parts of the ellipsoid produced by the plane (4) which lies on the same side of (4) as does the plane (7). On excluding from (9) its boundary circle we obtain the hemiellipsoid (8) into which the plane (7) is transformed.

It is now clear that the ellipsoid (10) has its center at O, has semimajor and semiminor axes each one unit long, and hence is an ellipsoid of revolution, symmetrical to the line m through O perpendicular to the plane (4). Since the hemiellipsoid (8) is also symmetrical to m, the length of the semimean axis is the unsigned distance from O to the point V where m pierces the hemiellipsoid. The equations of m are $x' = at$, $y' = bt$, $z' = ct$, where t denotes directed distance from O. Substitution in (8) gives

$$\frac{(a^2 + b^2 + c^2)t}{\sqrt{1 - (a^2 + b^2 + c^2)t^2}} = d \quad \text{or} \quad \frac{t}{\sqrt{1 - t^2}} = d,$$

so that t has the same sign as d. Hence

$$t = \frac{d}{\sqrt{1 + d^2}}$$

represents the directed distance OV, and the absolute value of this is the length of the semimean axis.

It has been shown that if p is any plane $ax+by+cz=d$, where $d \neq 0$ and $a^2+b^2+c^2=1$, it is transformed by T into a hemiellipsoid h whose shape, size, and position can be described as follows: (1) h is a surface of revolution interior to S, (2) the plane through O parallel to p meets S in the boundary of h, (3) h lies between this plane and p, (4) the line through O perpendicular to p is the axis of symmetry of h, (5) the point lying on this line, between O and p, and distant from O by $|d|/\sqrt{1+d^2}$, is the vertex V of h.

This description provides steps whereby h can be determined when p is given. Conversely, if a hemiellipsoid h interior to S is given whose boundary is a great circle of S and whose semimean axis OV has length $k<1$, there is a plane p which transforms into h. To determine it we extend OV beyond V to a point A such that $OA = k/\sqrt{1-k^2}$. The plane perpendicular to OA at A is p. In other

words, T establishes a 1-1 correspondence between all such hemiellipsoids and all planes not through O.

It is to be noted that a family of parallel planes is transformed into a single circular disc and all the hemiellipsoids "based" on it, that is, having a common boundary with it.

3. The effect of T on straight lines. Some agreements on terms will prove useful. We shall call the segment OV of the hemiellipsoid h its semimean axis, and by its semimajor and semiminor axes we shall mean the semimajor and semininor axes of the ellipsoid of which it is part. Each meridian curve on a hemiellipsoid is half of an ellipse whose major axis is a diameter of S and whose semiminor axis is the segment OV. We shall call the meridian curve a *semiellipse*, and by its semimajor and semiminor axes we shall mean the semimajor and semiminor axes of the ellipse.

Let g be any straight line in space. We first suppose it goes through O. Consider two planes containing g. These planes go through O, and hence each is transformed by T into that part of itself which is interior to S. Thus g is transformed into that part of itself which is interior to S, *i.e.*, a diameter of S exclusive of its endpoints. Conversely, each diameter of S, exclusive of its endpoints, is the transform of some straight line, *i.e.*, the line containing it.

Now suppose that g does not go through O. Let A be the foot of the perpendicular from O to g, π the plane through A perpendicular to OA, and π' the plane through g and OA. Then π will transform into a semiellipsoid h whose vertex V is on line OA, and π' into a circular disc D containing segment OV. Now g, being the intersection of π and π', transforms into the intersection of h and D. Since D goes through OP, the axis of symmetry of h, it must meet h in a meridian curve. Hence g transforms into a semiellipse. Conversely, let s be any semiellipse (exclusive of its endpoints) whose major axis is a diameter of S and whose semiminor axis OV is therefore of length $k<1$. Extend OV beyond V to a point A such that $OA=k/\sqrt{1-k^2}$, and let π be the plane through A perpendicular to OA. If π' is the plane of s, it follows that the line of intersection of π and π' is transformed into s.

Thus, all the straight lines in space transform into the diameters of S (excluding their endpoints) and the semiellipses (excluding their endpoints) having these diameters as major axes, and every such diameter or semiellipse is the transform of some straight line. A family of parallel lines goes into a single diameter and all semiellipses "based" on it, *i.e.*, having it as major axis. When a plane π through O is transformed into the corresponding circular disc, the straight lines of π go into the diameters of the disc and the semiellipses based on them. A family of parallel lines in π goes into a single diameter of the disc and the semiellipses based on it.

4. A topological model of Euclidean space. It may be helpful to summarize our main results. The interior of unit sphere S with center at O is a topological model M of Euclidean 3-space in which

(a) the points interior to S correspond to the points of space;

(b) the discs of unit radius with center at O and the hemiellipsoids based on them correspond to the planes of space (a single disc and the hemiellipsoids based on it corresponding to the family of planes parallel to the disc);

(c) the diameters of S and the semiellipses based on them correspond to the straight lines of space (a single diameter and the semiellipses based on it corresponding to the family of lines parallel to the diameter).

In particular, each disc is a topological model of a Euclidean plane in which a diameter of the disc and the semiellipses based on it correspond to the family of straight lines in the plane parallel to the diameter.

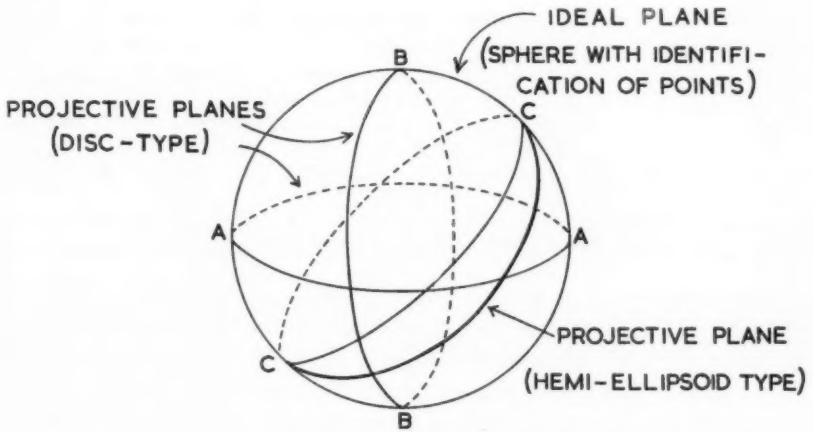

FIG. 1. Spherical-type model of projective 3-space.

5. A topological model of projective space. To the above model M of Euclidean space let us now add the points of S, that is, the points on the surface of S, and regard each two diametrically opposite points as identical. The resulting aggregate is a topological model M' of projective 3-space, for this addition of points is equivalent to the usual extension of Euclidean space by means of ideal points. Thus, with the specified identification of points, the surface of S and its great circles become a projective plane and its projective lines, corresponding to the ideal plane and its ideal lines. By the addition of the points of S the diameters and semiellipses of M are converted into projective lines, the discs and hemiellipsoids into projective planes (Fig. 1). A family in M consisting of a diameter and the semiellipses based on it (corresponding to a family of parallel straight lines in space) is converted into a family of projective lines on one point of S. A family in M consisting of a disc and the hemiellipsoids based on it (corresponding to a family of parallel planes in space) is converted into a family of

projective planes on one projective line of S. It is clear that the projective lines and planes of which we have been talking enjoy not only the foregoing properties but all of the other incidence relations possessed by the projective lines and planes of projective 3-space.

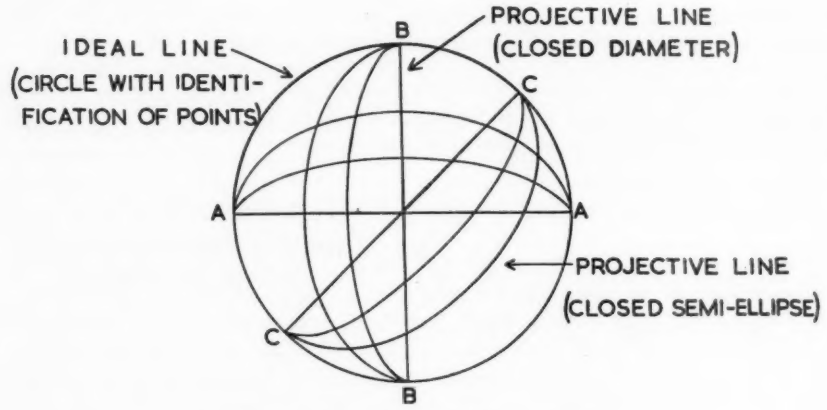

FIG. 2. Disc-type model of projective plane.

Incidentally, in adding to each disc of M its circular boundary we have obtained the two-dimensional projective model mentioned at the end of the introduction of this article. This is shown in Figure 2, where A, B, C are points of M.

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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

A METRIC FOR THE SPACE OF FUNCTION ELEMENTS

ANDREW M. GLEASON, Harvard University

When considering the problem of analytic continuation for functions of a complex variable, it is convenient to introduce the set $\mathcal{E}$ of all function elements, where a function element is a power series $\sum_{n=0}^{\infty} a_n(\zeta - z)^n$ with positive or

infinite radius of convergence R. $\mathcal{E}$ becomes a topological space when neighborhoods are defined in terms of analytic continuation and the connected components of $\mathcal{E}$ are the Riemann surfaces associated with complete analytic functions. From this fact it is clear that $\mathcal{E}$ is metrizable, but it is not clear that a metric can be easily defined. We shall exhibit a metric for $\mathcal{E}$ which has the very desirable property that the natural projection of $\mathcal{E}$ into the complex plane is a local isometry. Our metric will have the slight disadvantage that two points may be at distance ∞, but this does not vitiate any of the usual theorems about metric spaces; indeed, we can find a finite metric by setting $\rho'(a, b) = \min(1, \rho(a, b))$, but this new metric will not be quite as desirable as the old one.

By a function element we mean a sequence $e = \{z; a_0, a_1, \cdots\}$ of complex numbers where $\limsup |a_n|^{1/n} < \infty$. We identify e with the power series $\sum a_n(\zeta - z)^n$ which defines an analytic function (of ζ) in the open disk about z of radius $R(e) = \liminf |a_n|^{-1/n} (= \infty$ possibly); we shall refer to this disk as the e-disk. Let $\mathcal{E}$ be the set of all function elements.

Define complex valued functions on $\mathcal{E}$ by

$$p(e) = z,$$
$$F(e) = a_0, \qquad F'(e) = a_1, \cdots, F^{(n)}(e) = n!a_n, \cdots.$$

Here p is the natural projection of $\mathcal{E}$ onto the complex plane which takes a power series into its center and the functions $F, F', \cdots$ evaluate the function defined by a power series and its derivatives.

We shall say two function elements e_1 and e_2 are *adjacent* if their disks overlap, and in the overlapping region they define the same function. Since the intersection of two disks is always connected, in order that e_1 and e_2 be adjacent it is sufficient that they define functions which agree on some open set.

Define a function on $\mathcal{E} \times \mathcal{E}$ by

$$\rho(e_1, e_2) = R(e_1) + R(e_2) \qquad \text{if } e_1 \text{ and } e_2 \text{ are not adjacent,}$$
$$= |p(e_1) - p(e_2)| \qquad \text{if } e_1 \text{ and } e_2 \text{ are adjacent.}$$

This function takes extended nonnegative real values. We shall prove shortly that it is a metric for the set $\mathcal{E}$. It is entirely clear that ρ is symmetric and $\rho(e_1, e_2) = 0$ if and only if $e_1 = e_2$; hence we need only prove the triangle inequality. We note in passing that, in any event $\rho(e_1, e_2) \leqq R(e_1) + R(e_2)$, since when disks overlap their centers are closer than the sum of the radii.

LEMMA 1. *For all* $e_1, e_2 \in \mathcal{E}$, $R(e_1) + \rho(e_1, e_2) \geqq R(e_2)$.

Proof. There is nothing to prove unless $\rho(e_1, e_2) < R(e_2)$. In this case $p(e_1)$ falls in the e_2-disk because $\rho(e_1, e_2) = |p(e_1) - p(e_2)|$ and e_1 is obtained by developing the e_2-function about the point $p(e_1)$ since e_1 and e_2 are adjacent. The e_1-disk certainly contains the largest disk D with center $p(e_1)$ and contained in the e_2-disk. Since D has radius $R(e_2) - |p(e_1) - p(e_2)| = R(e_2) - \rho(e_1, e_2)$ we have $R(e_1) \geqq R(e_2) - \rho(e_1, e_2)$.

LEMMA 2. *Let D_1, D_2, D_3 be disks in the complex plane with centers z_1, z_2, z_3 and radii R_1, R_2, R_3, respectively. If $|z_1-z_2|<R_1+R_2$, $|z_2-z_3|<R_2+R_3$, and $|z_1-z_2|+|z_2-z_3|<R_1+R_3$, then $D_1\cap D_2\cap D_3$ is not void.*

Proof. This is trivial if $z_2\in D_1$ and $z_2\in D_3$. On the other hand, we cannot have both $z_2\notin D_1$ and $z_2\notin D_3$ since then $|z_1-z_2|\geqq R_1$ and $|z_2-z_3|\geqq R_3$ contradicting the third hypothesis. By symmetry, we may assume then, $z_2\in D_1$ but $z_2\notin D_3$. Consider the point x on the segment z_2z_3 such that $|x-z_3|=R_3$. We have $R_1+R_3>|z_1-z_2|+|z_2-z_3|=|z_1-z_2|+|z_2-x|+|x-z_3|$ so $R_1>|z_1-z_2|+|z_2-x|\geqq|z_1-x|$ and therefore $x\in D_1$. Moreover $|x-z_2|=|z_2-z_3|-R_3<R_2$ so $x\in D_2$. Since $x\in\overline{D}_3$ by construction and $D_1\cap D_2$ is a neighborhood of x, $D_1\cap D_2\cap D_3$ is not void.

We are now in a position to prove the triangle inequality for ρ; that is, $\rho(e_1, e_2)+\rho(e_2, e_3)\geqq\rho(e_1, e_3)$. There are three cases. If e_2 is not adjacent to e_1, then $\rho(e_1, e_2)+\rho(e_2, e_3)=R(e_1)+R(e_2)+\rho(e_2, e_3)\geqq R(e_1)+R(e_3)\geqq\rho(e_1, e_3)$ using Lemma 1, and similarly if e_2 is not adjacent to e_3. Finally we consider the possibility that e_2 is adjacent to both e_1 and e_3. In that case $\rho(e_1, e_2)=|p(e_1)-p(e_2)|$ and $\rho(e_2, e_3)=|p(e_2)-p(e_3)|$, so if $|p(e_1)-p(e_2)|+|p(e_2)-p(e_3)|\geqq R(e_1)+R(e_3)$ the required result follows immediately. On the other hand, if $|p(e_1)-p(e_2)|+|p(e_2)-p(e_3)|<R(e_1)+R(e_3)$, then Lemma 2 applies to the e_1-, e_2-, and e_3-disks and we conclude that these disks mutually overlap. Since both e_1 and e_3 agree with e_2 on the common part, e_1 and e_3 are adjacent so $\rho(e_1, e_3)=|p(e_1)-p(e_3)|$ and the required inequality is the familiar triangle inequality in the complex plane.

The metric ρ defines a topology for the set $\mathcal{E}$. The following facts about the space $\mathcal{E}$ are obvious from the definition of ρ. The projection p of $\mathcal{E}$ onto the complex plane carries the ball about e of radius $R(e)$ homeomorphically onto the e-disk and even maps the ball of radius $\frac{1}{2}R(e)$ isometrically, for any two elements in the latter ball are adjacent. The functions $F^{(n)}$ are all continuous since locally at e they can be represented in the form $f^{(n)}\circ p$ where f is the function defined by e. The function R is continuous since $|R(e_1)-R(e_2)|\leqq\rho(e_1, e_2)$ by Lemma 1. From these facts it is clear that ρ defines the usual topology for $\mathcal{E}$.

SECOND DERIVATIVES ON LEVEL SURFACE ELEMENTS

SIDNEY F. MACK, The Pennsylvania State University

Let $F(x, y, z)$ denote a function of class C_1 in some domain D in space which is constant on a level surface element S contained in D. It is well known that at every interior point P of S the first derivative of F in the direction tangent to S at P vanishes. In this paper, with F of class C_2, we give some formulas for the second derivatives on level surfaces, which show how the derivatives are related to the geometric properties of the surfaces. In the following, the directions x and y are chosen tangent to S at P, k_x is the normal curvature of S at P in the x direction, T_x is the geodesic torsion of S at P in the x direction, K is the mean curvature of S at P, and ∇^2F is the Laplacian of F.

THEOREM. *Let F be of class C_2 in a domain D in space and let S be a level surface element of F which is contained in D. With x, y, n denoting a tangent-normal system of rectangular coordinates at the interior point P of S, we have*

$$\frac{\partial^2 F}{\partial x^2} = -k_x \frac{\partial F}{\partial n}, \tag{1}$$

$$\frac{\partial^2 F}{\partial x \partial y} = T_x \frac{\partial F}{\partial n}, \tag{2}$$

$$\frac{\partial^2 F}{\partial n^2} = 2K \frac{\partial F}{\partial n} + \nabla^2 F. \tag{3}$$

Proof. Let the surface element S be given by $z=f(x, y)$ with reference to the coordinates x, y in the tangent plane at P, and let α, β, ζ denote a tangent-normal system of rectangular coordinates at the arbitrary point Q of S. Since F is constant on S and S is of class C_2, we find that $(\partial F/\partial \alpha)_Q = (\partial F/\partial \beta)_Q = 0$. From this, the general expression

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \alpha} \cos(x, \alpha) + \frac{\partial F}{\partial \beta} \cos(x, \beta) + \frac{\partial F}{\partial \zeta} \cos(x, \zeta)$$

reduces on S to

$$\left(\frac{\partial F}{\partial x}\right)_Q = \left(\frac{\partial F}{\partial \zeta}\right)_Q \cos(x, \zeta), \qquad Q \in S. \tag{4}$$

Let M be a point of the x-axis and Q its projection on S parallel to the z-axis. Then

$$\left(\frac{\partial F}{\partial x}\right)_Q = \left(\frac{\partial F}{\partial x}\right)_M + \overline{MQ} \left(\frac{\partial^2 F}{\partial x \partial z}\right)_{M'}, \tag{5}$$

where M' is some point between M and Q.

Now, since $(\partial F/\partial x)_P = 0$, we have

$$\left(\frac{\partial^2 F}{\partial x^2}\right)_P = \lim_{M \to P} \frac{1}{\overline{PM}} \left[\left(\frac{\partial F}{\partial x}\right)_M - \left(\frac{\partial F}{\partial x}\right)_P\right] = \lim_{M \to P} \frac{1}{\overline{PM}} \left(\frac{\partial F}{\partial x}\right)_M.$$

Making use of (4) and (5) and the fact that $\overline{MQ}$ vanishes like $\overline{PM}^2$ while $\partial^2 F/\partial x \partial z$ remains bounded, we arrive at

$$\left(\frac{\partial^2 F}{\partial x^2}\right)_P = \lim_{M \to P} \frac{1}{\overline{PM}} \left(\frac{\partial F}{\partial \zeta}\right)_Q \cos(x, \zeta).$$

But,

$$\lim_{M \to P} \frac{\cos(x, \zeta)}{\overline{MP}} = \left(\frac{\partial}{\partial x} \cos(x, \zeta)\right)_P$$

which, in view of

$$\cos(x, \zeta) = \frac{-(\partial f/\partial x)_Q}{\sqrt{1 + (\partial f/\partial x)_Q^2 + (\partial f/\partial y)_Q^2}},$$

$$\left(\frac{\partial f}{\partial x}\right)_P = \left(\frac{\partial f}{\partial y}\right)_P = 0, \qquad \left(\frac{\partial^2 f}{\partial x^2}\right)_P = k_x(P),$$

yields formula (1).

The derivation of the second formula follows from analogous considerations; we use here the relation $T_x(P) = -(\partial^2 f/\partial x \partial y)_P$. The third formula is obtained directly from Laplace's operator, upon making use of formulas for $(\partial^2 F/\partial x^2)_P$ and $(\partial^2 F/\partial y^2)_P$ in addition to the relation $k_x(P) + k_y(P) = 2K(P)$.

The tangent-normal derivative $\partial^2 F/\partial x \partial n$ does not seem to be related to the second-order geometric properties of the level surface element.

The author is greatly indebted to Professor Griffith C. Evans for suggesting the method employed in obtaining the formulas in this paper.

ON THE CAUCHY-LIPSCHITZ THEOREM

GIDEON PEYSER, Pratt Institute, Brooklyn

Our purpose is to present a proof of the classical Cauchy-Lipschitz theorem concerning the uniqueness and existence of the solution of the ordinary differential equation of first order. Our methods differ from well-established classical procedures. These methods lend themselves to various other fields in analysis. Uniqueness is proved by solving an integral inequality. For the existence we use the usual Picard iteration but introduce an exponential contracting factor to prove convergence. We also use the described methods to obtain an *a priori* estimate of the solution.

The Cauchy-Lipshitz theorem states:

If $f(x, y)$ is defined in the rectangle R, $x_0 \leqq x \leqq x_0 + a$, $y_0 - b \leqq y \leqq y_0 + b$, continuous in R with respect to x and satisfies the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leqq A|y_1 - y_2|$$

for all pairs y_1, y_2 uniformly in x, then there exists a uniquely determined function $y(x)$ with $y(x_0) = y_0$ satisfying the differential equation

$$dy/dx = f(x, y) \tag{1}$$

in the interval $x_0 \leqq x \leqq x_0 + h$ with $h = \min(a, b/M)$ where $M = \max_{(x,y)\in R} |f(x, y)|$.

Proof of uniqueness. If $u(x)$ and $w(x)$ are two solutions of (1) with $u(x_0) = w(x_0) = y_0$, then

$$|u(x) - w(x)| \leqq \int_{x_0}^{x} |f(\xi, u(\xi)) - f(\xi, w(\xi))|\, d\xi \leqq A \int_{x_0}^{x} |u(\xi) - w(\xi)|\, d\xi. \tag{2}$$

Denoting the right hand side of (2) by $W(x)$, we have $W(x) \geqq 0$ and $W(x_0) = 0$ and $dW/dx \leqq AW$.

Therefore $e^{-Ax}dW/dx - Ae^{-Ax}W \leqq 0$ and $d/dx(e^{-Ax}W) \leqq 0$. Integrating from x_0 to x we have $0 \leqq e^{-Ax}W \leqq 0$ and therefore $W \equiv 0$. From (2) follows $u(x) \equiv w(x)$.

Proof of existence. Following the classical Picard iteration, we define

$$
\begin{aligned}
y_0(x) &= y_0, \\
y_{n+1}(x) &= y_0 + \int_{x_0}^{x} f(\xi, y_n(\xi))d\xi.
\end{aligned} \tag{3}
$$

The restriction $h \leqq b/M$ assures us that all operations take place in R and therefore the process can be carried out.

To establish uniform convergence of the iteration process we proceed as follows:

$$
|y_{n+1} - y_n| \leqq \int_{x_0}^{x} |f(\xi, y_n) - f(\xi, y_{n-1})|\, d\xi \leqq A \int_{x_0}^{x} |y_n - y_{n-1}|\, d\xi. \tag{4}
$$

Multiply both sides of (4) by $e^{-\lambda x}$ where λ is a positive constant at our disposal. Integrate both sides from x_0 to x using integration by parts on the right. We get

$$
\int_{x_0}^{x} e^{-\lambda\xi} |y_{n+1} - y_n|\, d\xi \leqq A/\lambda \int_{x_0}^{x} e^{-\lambda\xi} |y_n - y_{n-1}|\, d\xi. \tag{5}
$$

Repeating the same argument with the right side of (5) we have, choosing λ such that $A/\lambda = q < 1$,

$$
\int_{x_0}^{x} e^{-\lambda\xi} |y_{n+1} - y_n|\, d\xi \leqq q^n \int_{x_0}^{x} e^{-\lambda\xi} |y_1 - y_0|\, d\xi \leqq Bq^n.
$$

Therefore

$$
\begin{aligned}
|y_{n+1} - y_n| &\leqq \int_{x_0}^{x} |f(\xi, y_n) - f(\xi, y_{n-1})|\, d\xi \leqq A \int_{x_0}^{x} |y_n - y_{n-1}|\, d\xi \\
&\leqq Ae^{\lambda(x_0+h)} \int_{x_0}^{x} e^{-\lambda\xi} |y_n - y_{n-1}|\, d\xi \leqq Cq^{n-1}.
\end{aligned} \tag{6}
$$

Therefore

$$
|y_{n+m} - y_n| \leqq |y_{n+m} - y_{n+m-1}| + \cdots + |y_{n+1} - y_n| \leqq Cq^{n-1}/(1-q)
$$

and $y_n(x)$ approaches uniformly a continuous function $y(x)$ which satisfies

$$
y(x) = y_0 + \int_{x_0}^{x} f(\xi, y(\xi))d\xi.
$$

This completes the existence proof.

A proof similar to the uniqueness proof yields an estimate of the solution of equation (1):

$$|y(x) - y_0| \leqq e^{Ax} \int_{x_0}^{x} e^{-A\xi} |f(\xi, y_0)| \, d\xi \qquad x_0 \leqq x \leqq x_0 + h. \tag{7}$$

Proof.

$$\begin{aligned} |y(x) - y_0| &\leqq \int_{x_0}^{x} |f(\xi, y(\xi))| \, d\xi \\ &\leqq \int_{x_0}^{x} |f(\xi, y(\xi)) - f(\xi, y_0)| \, d\xi + \int_{x_0}^{x} |f(\xi, y_0)| \, d\xi \\ &\leqq A \int_{x_0}^{x} |y(\xi) - y_0| \, d\xi + \int_{x_0}^{x} |f(\xi, y_0)| \, d\xi. \end{aligned} \tag{8}$$

Denoting the right hand side of (8) by $V(x)$, we have $V(x) \geqq 0$ and $V(x_0) = 0$ and $dV/dx - |f(x, y_0)| \leqq AV$. Therefore $e^{-Ax} dV/dx - Ae^{-Ax} V \leqq e^{-Ax} |f(x, y_0)|$ and $dV/dx(e^{-Ax} V) \leqq e^{-Ax} |f(x, y_0)|$. Integrating from x_0 to x,

$$e^{-Ax} V \leqq \int_{x_0}^{x} e^{-A\xi} |f(\xi, y_0)| \, d\xi$$

and from (8) follows

$$|y(x) - y_0| \leqq e^{Ax} \int_{x_0}^{x} e^{-A\xi} |f(\xi, y_0)| \, d\xi.$$

It is clear that the above methods can also be used for systems of first order ordinary differential equations.

NOTE ON THE CONVERGENCE OF SERIES OF ULTRASPHERICAL POLYNOMIALS

D. P. Gupta, University of Saugar, Sagar, India

1. For the series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \tag{1.1}$$

Kolmogoroff ([1], p. 109) has proved the following theorem:

If $a_n \to 0$ and the sequence $\{a_n\}$ is quasiconvex, the series (1.1) *converges, save for $x = 0$, to an integrable function $f(x)$, and is the Fourier series of $f(x)$.*

Zitarosa [2] has proved a similar theorem for the series of Legendre polynomials. In the present note we establish the following more general result, which includes Zitarosa's theorem as a particular case for $\lambda = 1/2$.

THEOREM. *If* $a_n = o(n^\mu)$ *and the sequence*

$$\left\{\frac{a_n}{(n+\lambda)^\mu}\right\} \tag{1.2}$$

is of bounded variation for $0<\mu<\lambda$ *and* $0<\lambda\leqq 1/2$, *then the series*

$$\sum_{n=0}^{\infty} a_n P_n^{(\lambda)}(x) \tag{1.3}$$

converges uniformly in the interval $(-1+\delta, 1-\delta)$, $P_n^{(\lambda)}(x)$ *being the ultraspherical polynomial of order* λ.

In view of Exercise 1 ([1], p. 129), the above theorem can be stated in the following form:

If $\{a_n/(n+\lambda)^\mu\}$ *tends to* 0 *and is quasiconvex for* $0<\mu<\lambda$ *and* $0<\lambda\leqq 1/2$, *then the series* (1.3) *is uniformly convergent in the range* $(-1+\delta, 1-\delta)$.

2. *Proof.* From Szegö ([3], p. 82) we have

$$\sum_{\nu=0}^{n} (\nu+\lambda)P_\nu^{(\lambda)}(x) = \frac{1}{2}\,\frac{(n+2\lambda)P_n^{(\lambda)}(x) - (n+1)P_{n+1}^{(\lambda)}(x)}{1-x}. \tag{2.1}$$

Also, Szegö ([3], p. 166) has proved that for $0<\lambda\leqq 1/2$ and $\epsilon\leqq\theta\leqq\pi-\epsilon$,

$$|P_n^{(\lambda)}(\cos\theta)| < A(\epsilon)n^{\lambda-1} \tag{2.2}$$

where $A(\epsilon)$ denotes a constant for a fixed $\epsilon>0$, not necessarily the same at each occurrence.

Hence,

$$\begin{aligned}\left|\sum_{\nu=0}^{n} (\nu+\lambda)P_\nu^{(\lambda)}(x)\right| &< \frac{1}{2}\,\frac{(n+2\lambda)|P_n^{(\lambda)}(x)| + (n+1)|P_{n+1}^{(\lambda)}(x)|}{1-x} \\ &< \frac{1}{2}A(\epsilon)\left\{\frac{(n+2\lambda)n^{\lambda-1} + (n+1)^\lambda}{1-x}\right\} \qquad (2.3)\\ &< A(\epsilon)\cdot n^\lambda.\end{aligned}$$

Consider now the series

$$\sum_{\nu=0}^{n} (\nu+\lambda)^\mu P_\nu^{(\lambda)}(x) \qquad \text{for } \mu<\lambda.$$

Let

$$\sum_{\nu=0}^{n} (\nu+\lambda)^\mu P_\nu^{(\lambda)}(x) = \sum_{\nu=0}^{n} c_\nu\phi_\nu(x),$$

where $c_\nu = (\nu+\lambda)^{\mu-1}$ and $\phi_\nu(x) = (\nu+\lambda)P_\nu^{(\lambda)}(x)$. Write

$$\Phi_n(x) = \sum_{\nu=0}^{n} \phi_\nu(x) = \sum_{\nu=0}^{n} (\nu+\lambda)P_\nu^{(\lambda)}(x).$$

Using Abel's transformation, we have

$$\left|\sum_{\nu=0}^{n} c_\nu\phi_\nu(x)\right| = \left|\sum_{\nu=0}^{n-1} \Phi_\nu(x)\{(\nu+\lambda)^{\mu-1} - (\nu+\lambda+1)^{\mu-1}\} + \Phi_n(x)(n+\lambda)^{\mu-1}\right|$$

$$\leqq \sum_{\nu=0}^{n-1} |\Phi_\nu(x)|\,|(\nu+\lambda)^{\mu-1} - (\nu+\lambda+1)^{\mu-1}| + |\Phi_n(x)|\,|(n+\lambda)^{\mu-1}|$$

$$< A(\epsilon)\left\{\sum_{\nu=0}^{n-1} \nu^\lambda |(\nu+\lambda)^{\mu-1} - (\nu+\lambda+1)^{\mu-1}| + n^\lambda(n+\lambda)^{\mu-1}\right\} \tag{2.4}$$

$$< A(\epsilon)\left\{\sum_{\nu=0}^{\infty} \nu^\lambda(\nu+\lambda)^{\mu-2}\cdot B + O(1)\right\} \qquad (B \text{ being a constant}).$$

$$< A(\epsilon).$$

(1.3) is

$$\sum_{n=0}^{\infty} a_n P_n^{(\lambda)}(x) = \sum_{n=0}^{\infty} \frac{a_n}{(n+\lambda)^\mu}(n+\lambda)^\mu P_n^{(\lambda)}(x)$$

$$= \sum_{n=0}^{\infty} b_n\{c_n\phi_n(x)\}, \text{ say.}$$

The sequence $\{b_n\}\to 0$ and is of bounded total fluctuation, and $\sum c_n\phi_n(x)$ has its partial sums uniformly bounded in view of (2.4). Thus the theorem is proved by virtue of Abel's criterion for uniform convergence of series.

It should be interesting to solve the corresponding problem for Jacobi polynomials. I am thankful to Dr. B. N. Prasad for his kind help in the preparation of this note.

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CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

A CALCULUS PROBLEM WITH OVERTONES IN RELATED FIELDS

C. S. OGILVY, Hamilton College

Several elementary extremum problems in the calculus which seem at first glance to be quite distinct from one another can be linked together through considerations not usually presented in the texts.

I. One ship, A, is anchored 9 miles offshore. Opposite a point 6 miles down the coast a second ship, B, is anchored 3 miles offshore. A boat from A is to put a passenger ashore and then proceed to B (Fig. 1).

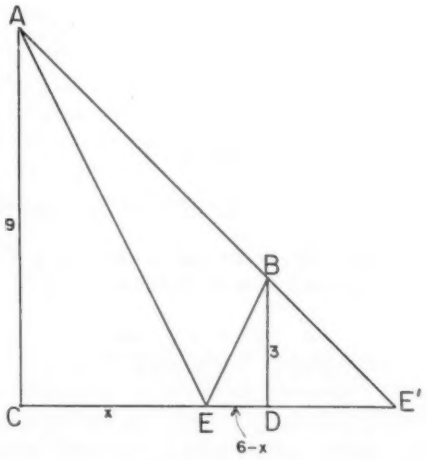

FIG. 1

(a) What is the shortest course for the boat?

(b) Suppose the passenger pays each member of the boat's crew \$1 per mile for every mile they carry him; but each member of the crew must forfeit \$1 for every mile travelled from shore to B. What is the course which will yield them the maximum profit, under the restriction that it must consist of two straight-line segments?

II. A man can row 4 miles per hour and run 5 miles per hour. He leaves ship B of problem I wishing to go to point Q on the shore 5 miles from D (Fig. 2).

(a) Where should he land in order to reach Q most quickly?

These do not look to the student like connected problems. I(a) seeks a minimum distance, I(b) a maximum profit, and II(a) a minimum time. We observe first that the two (a) parts are both solved by applying Fermat's principle of optics, where in I the shore is a reflector for light travelling from A to B and in II the shore is a refractor with light travelling from the slower medium into the faster. From this point of view, both problems do minimize the same variable, time. In the first instance, the physicist seeks two similar triangles so that the angle of incidence will equal the angle of reflection: $9/x=3/(6-x)$, or $x=CE=4.5$ miles. In II(a) the light must travel in conformance with Snell's Law: $\sin\alpha/\sin\beta=4/5$. But this is the limiting case where $\beta=\pi/2$, which means $\sin\alpha=4/5$, or $DS=4$ miles.

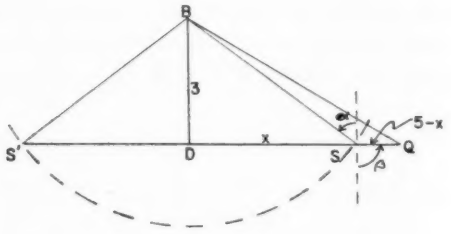

FIG. 2

We turn now to the calculus method of handling these two problems.

I(a). The distance function to be minimized is $\sqrt{81+x^2}+\sqrt{9+(6-x)^2}$. Setting the first derivative equal to zero gives immediately the equation

$$\frac{x}{\sqrt{81+x^2}}=\frac{6-x}{\sqrt{9+(6-x)^2}}. \tag{1}$$

The "solutions" of (1) are $x=4.5$, 9. The student throws away the second value because he feels pretty certain that $x<6$. This is the wrong reason for throwing it away. The function is not stationary at $x=9$ and indeed 9 does not satisfy (1). It is purely extraneous, introduced by squaring in the course of solving (1). But although "extraneous" roots may have no place in algebra, there is usually some reason for their appearance in the calculus. We now discover the meaning of this one.

I(b). The profit function to be maximized is $\sqrt{81+x^2}-\sqrt{9+(6-x)^2}$, which leads, on differentiation, to

$$\frac{x}{\sqrt{81+x^2}}=\frac{x-6}{\sqrt{9+(6-x)^2}}. \tag{2}$$

Observe that when we square both sides of (2) it is identical with the equation resulting from squaring both sides of (1). Hence solutions again appear to be 4.5 and 9; but this time 9 satisfies (2) and 4.5 does not. The crew should land the passenger at E'.

At this stage we can formulate a General Remark:

If two problems requiring different solutions lead at some stage to the same equation, then this equation must deliver at least two solutions, one of which will be extraneous to each problem.

II(a). The time function to be minimized is $\sqrt{9+x^2}/4+(5-x)/5$. Setting the first derivative equal to zero we get

$$\frac{x}{\sqrt{9+x^2}} = \frac{4}{5}, \tag{3}$$

or $x = \pm 4$. As before, only one of the two values of x is admissible. It is $+4$ which satisfies (3). And now our General Remark *predicts* the existence of a companion problem:

II(b). Suppose the man has to pay the owner of the boat \$1 for each hour he rows, but some altruistic soul pays him \$1 per hour for each hour he runs. What now is his most profitable course, assuming still that he runs only straight along the shore after landing?

The profit function to be maximized is $-\sqrt{9+x^2}/4+(5-x)/5$, which yields

$$\frac{x}{\sqrt{9+x^2}} = -\frac{4}{5}, \tag{4}$$

and only -4 satisfies (4). He should row to S'.

These results have interesting geometric interpretations. Equations (1) and (2) are proportions calling for similar right triangles. There are *two* pairs of similar right triangles formed by the ships and the shoreline, and E and E' give them both. Equations (3) and (4) ask for $\alpha = \sin^{-1}(\pm 4/5) = \cos^{-1}(3/5)$. This is the "ambiguous case" of high school trigonometry: triangles BQS and BQS' both satisfy the required conditions. Student reaction to these unexpected cross-connections with previously studied topics has been favorable.

We leave to the reader the statement of the more difficult problems III(a) and III(b) where the point Q is inland. Snell's Law this time provides no short-cut, and the solution is technically burdensome because the terms in x^3 and x^4 do not drop out in either the physics or the calculus method. It is believed that only two of the roots of the resulting equation are ever real, namely, the answers to III(a) and III(b) respectively.

I am indebted to Professor H. M. Gehman of the University of Buffalo for the statement of problem I(b).

A THEOREM ON COLLEGE ALGEBRA

THOMAS E. MOTT, The Pennsylvania State University

The following theorem in college algebra has constituted a part of the course in freshman mathematics as I have taught it at the Pennsylvania State University for the past few years. This theorem, a complete statement and proof of which seem to be missing from the texts which I have examined, is one of the simplest tests for the upper and lower bounds of the roots of a polynomial equation.

Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ be a polynomial with integral coefficients and r be a real number. Define, for each nonnegative integer $k\leqq n$, $b_k=a_nr^{n-k}+a_{n-1}r^{n-k-1}+\cdots+a_k$.

THEOREM.

(A) *If the b_k are all of the same sign and $r\geqq 0$, then no $x>r$ satisfies $f(x)=0$.*

(B) *If the numbers b_k alternate in sign and $r\geqq 0$, then no $x<0$ satisfies $f(x)=0$.*

(C) *If the numbers b_k alternate in sign and $r\leqq 0$, then no $x<r$ satisfies $f(x)=0$.*

Proof. Consider the following diagram which appears when synthetic division is used:

$$\begin{array}{cccccc} a_n & a_{n-1} & \cdots & a_1 & a_0 & \big|\, r \\ & rb_n & & rb_2 & rb_1 & \\ \hline b_n & b_{n-1} & \cdots & b_1 & b_0 & \end{array}.$$

From this diagram, we see that $b_n=a_n$, $b_0=f(r)$, and

$$f(x) = (x-r)(b_nx^{n-1}+b_{n-1}x^{n-2}+\cdots+b_2x+b_1)+b_0.$$

(A) $f(x)=(x-r)(b_nx^{n-1}+\cdots+b_1)+b_0$ and for $x>r$ the factor $(x-r)$ is positive. If the signs of the b_k are all positive we have $f(x)>b_0>0$ for $x>r\geqq 0$, whereas if the signs of the b_k are all negative we have $f(x)<b_0<0$ for $x>r\geqq 0$.

(B) $f(x)=(x-r)(b_nx^{n-1}+\cdots+b_1)+b_0$ and since $r\geqq 0$ the factor $(x-r)$ is negative for $x<0$. Since the signs of the b_k alternate, the expression $(x-r)\cdot(b_nx^{n-1}+\cdots+b_1)$ will have the same sign as b_0 for $x<0$. Thus for $x<0$ we have $f(x)>b_0>0$ if $b_0>0$ and $f(x)<b_0<0$ if $b_0<0$.

(C) $f(x)=(x-r)(b_nx^{n-1}+\cdots+b_1)+b_0$ and for $x<r$ the factor $x-r$ is negative. Since the signs of the b_k alternate, the expression $(x-r)(b_nx^{n-1}+\cdots+b_1)$ will have the same sign as b_0 for $x<r\leqq 0$. Thus for $x<r\leqq 0$ we have $f(x)>b_0>0$ if $b_0>0$ and $f(x)<b_0<0$ if $b_0<0$.

One might now ask whether we may associate a convenient plus or minus sign with any zero entry along the bottom row in synthetic division. We need only observe that $b_n=a_n\neq 0$ and the proofs all remain unchanged, with the exception that we now must write $f(x)>b_0\geqq 0$ and $f(x)<b_0\leqq 0$ in place of $f(x)>b_0>0$ and $f(x)<b_0<0$ respectively in order to allow for the possibility that $b_0=0$.

In order to see in (A) that we cannot prove any general result for $r<0$, con-

sider the equation $x^2+2x+1=0$ with $r=-3/2$. In (B), the equation $x^2-5x+6=0$ with $r=4$ shows that we cannot conclude that roots will not exist in the interval $0 \leqq x < r$.

THE RAZOR'S EDGE?

C. S. Ogilvy, Hamilton College and N. G. Gunderson, University of Rochester

Most modern calculus texts are careful to point out that the vanishing of the first partial derivatives is a necessary *but not sufficient* condition for a maximum or minimum of a function of two variables. One must examine the second partials also. Failure to do so has led more than one writer to include in his textbook an apparently plausible problem which in fact has a strange solution.* Interestingly enough the same problem, with only slight modifications, has found its way into several of the prominent texts.

In one book the problem is stated as follows: "A manufacturer produces safety razors and blades at a cost of 20¢ per razor and 10¢ per dozen blades. If he charges x cents per razor and y cents per dozen blades, he finds that he can sell $100{,}000/xy$ razors and $400{,}000/xy$ dozen blades daily. How should he fix prices so as to maximize his profit?"

The profit z is readily expressed as a function of x and y: $z = f(x, y) = 100{,}000\ (1/y+4/x-60/xy)$. If one equates the two first partials to zero and solves the resulting two equations simultaneously, one arrives quite happily at $x=60$, $y=15$. These apparently reasonable figures yield a profit of \$66.67. But a closer examination of the profit function discloses some disturbing facts. The ordinary three-dimensional curved surface represented by $z=f(x, y)$ has negative Gaussian curvature at the point P: $(60, 15, 6666\frac{2}{3})$; that is, the surface is saddle-shaped at P, so that although the first partial derivatives are both zero, every neighborhood of P contains points where z has a greater value and points where z has a lesser value than at P. Furthermore the surface has two horizontal rulings through P, straight lines lying in the surface, one parallel to the X-axis and the other parallel to the Y-axis. This means that so long as $x=60$, z is independent of y, and if $y=15$, z is independent of x. The company could sell the blades for 15¢ a dozen and the razors at a penny apiece and make the same profit.

But we have worse in store. Let the blades be sold for more than 15¢ per dozen, and it matters not how little more—say at 16¢ per dozen. Now by selling the razors at a price arbitrarily close to zero, the profits can be made to soar indefinitely! We are drawn to the reluctant conclusion that there must be something wrong with this economic dream.

The trouble, of course, lies in the original data. The immediate concern of our manufacturer is not how best to fix prices, but how most quickly to put his sales

* This was initially called to our attention by Dr. Alfred W. Jones of Bell Telephone Laboratories.

manager on the carpet and try to find out where he dredged up this formula. No law of supply and demand ever behaved that way. The price of each commodity is varying inversely as the product of both together. This says that a change in the price of only one affects the sales of both of them in the same ratio. Such a relation can be realistically interpreted in only one way: no one buys razors without blades or blades without razors; and each sale of one razor carries with it the automatic sale of some fixed number of blades, and vice-versa. This is another way of saying that the price of the package is to be treated as a single variable. And if indeed one lets $y=kx$, so that the profit reduces to a function of one variable, the problem begins to make sense. Even so, however, $x=60$ and $y=15$ is not the best solution. It is the optimum for $k=\frac{1}{4}$; but because P is a saddle point *any* other k will produce a maximum profit higher than \$66.67. Now where did that sales manager disappear to?

The problem can be salvaged. A corrected version appears in a revised edition of one of the texts which formerly contained the faulty problem. In the revised problem, $100{,}000/xy$ is replaced by $100{,}000/x^2y$, and $400{,}000/xy$ is replaced by $400{,}000/xy^2$. Everything else remains the same. The surface represented by the new $z=f(x, y)$ has positive curvature in the region in question and a true maximum at the point $x=12$, $y=24$, an elliptic point. We now have the unexpected result that the razors are being sold below cost. This is not inconsistent; that it can happen to certain items of a line is well known to all manufacturers. But there is another objection. In the revised problem both the retail prices and the daily profit are too low to be realistic. If we write 1,000,000 in place of both the 100,000 and the 400,000 in the statement of the revised problem, we end up with more reasonable figures throughout. We leave the working of this final version to the reader or his calculus class.

A DEFINITE INTEGRAL

ALBERT WILANSKY, Lehigh University

To evaluate $\int_a^b x^p dx$ without use of the fundamental theorem of calculus, textbooks usually assume that the integral exists (appealing, say, to a general existence theorem) and then choose a definite sequence of partitions, using finally, in some cases, an identity involving $\sum k^p$. The following derivation will give the existence and value of the integral without assuming the existence first. In addition, an inequality, (3), is given connecting the integral and the Riemann sum for an arbitrary partition. The case $p=1$ is exceptionally easy and well motivated.

We assume that p is a positive integer. We begin with the identity

$$B^p(B-A)=\frac{B^{p+1}-A^{p+1}}{p+1}+(B-A)\left[B^p-\frac{B^p+B^{p-1}A+B^{p-2}A^2+\cdots A^p}{p+1}\right].$$

A glance at the right-hand side verifies this. A suggestion for motivating it is

as follows: We consider a typical interval $[x_{k-1}, x_k]$, call it $[A, B]$. Then $B^p(B-A)$ is the area of a rectangle, $(B^{p+1}-A^{p+1})/(p+1)$ is the area under the curve $y=x^p$, and the remaining term is the area of the curvilinear triangle above the curve. This is no proof, of course. The identity is proved by checking it.

If we assume $A<B$, the quantity inside square brackets is less than

$$B^p - \frac{A^p + A^p + \cdots + A^p}{p+1} = B^p - A^p.$$

Thus we have

$$(1) \qquad B^p(B-A) < \frac{B^{p+1} - A^{p+1}}{p+1} + (B-A)(B^p - A^p) \qquad \text{if } A < B.$$

Similarly

$$(2) \qquad A^p(B-A) > \frac{B^{p+1} - A^{p+1}}{p+1} - (B-A)(B^p - A^p) \qquad \text{if } A < B.$$

Next consider a partition of $[a, b]$, $a=x_0<x_1<x_2 \cdots <x_n=b$, $x_{k-1} \leqq \xi_k \leqq x_k$, and the Riemann sum $\sum f(\xi_k)(x_k-x_{k-1}) = \sum \xi_k^p(x_k-x_{k-1})$ which we shall denote by $\sum$. Clearly

$$\sum x_{k-1}^p(x_k - x_{k-1}) \leqq \sum \leqq \sum x_k^p(x_k - x_{k-1}).$$

Apply (1), (2) with $A=x_{k-1}$, $B=x_k$; this yields

$$\frac{1}{p+1}\sum (x_k^{p+1} - x_{k-1}^{p+1}) - \sum (x_k - x_{k-1})(x_k^p - x_{k-1}^p)$$

$$\leqq \sum \leqq \frac{1}{p+1}\sum (x_k^{p+1} - x_{k-1}^{p+1}) + \sum (x_k - x_{k-1})(x_k^p - x_{k-1}^p),$$

i.e.,

$$\frac{x_n^{p+1} - x_0^{p+1}}{p+1} - \delta \sum (x_k^p - x_{k-1}^p) \leqq \sum \leqq \frac{x_n^{p+1} - x_0^{p+1}}{p+1} + \delta \sum (x_k^p - x_{k-1}^p),$$

where δ is the norm of the partition $(=\max (x_k-x_{k-1}))$. Hence

$$\frac{b^{p+1} - a^{p+1}}{p+1} - \delta(b^p - a^p) \leqq \sum \leqq \frac{b^{p+1} - a^{p+1}}{p+1} + \delta(b^p - a^p).$$

This yields

$$(3) \qquad \left| \sum - \frac{b^{p+1} - a^{p+1}}{p+1} \right| \leqq \delta(b^p - a^p)$$

and the result follows.

MATHEMATICAL EDUCATION NOTES

Edited by JOHN R. MAYOR, American Association for the Advancement of Science and the University of Maryland, and JOHN A. BROWN, University of Delaware

Contributions for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

NOTE ON DEGREES GRANTED IN 1956–57

A report issued recently by the Office of Education shows the number of bachelor's degrees in mathematics granted by 1,341 institutions of higher education in 1956–57 was 5,546, an increase of 19 per cent over the previous year. The total number of bachelor's degrees conferred in all fields showed an increase of 9 per cent for this period. The per cent increase in the other sciences was greater than the per cent increase in number of degrees in all fields but in no case as great as in mathematics. Mathematics was the only one of the sciences in which more than half (52.1 per cent) of the bachelor's degrees in 1956–57 were granted by publicly-controlled institutions as compared with privately-controlled institutions.

The report also includes data on second-level (master's) and doctorate degrees. Numbers of degrees at the three levels are tabulated by states and institutions of higher education within the states. The state and institutional tabulation is broken down into major fields of study. The 196-page report, prepared by the U. S. Department of Health, Education, and Welfare, Office of Education, can be obtained from the U. S. Government Printing Office, Washington 25, D. C. for $1.50.

SUNRISE SEMESTER AT NEW YORK UNIVERSITY*

WCBS-TV and New York University this fall will expand "Sunrise Semester," the early-morning educational television program, to include four college-credit courses. Beginning September 29, "Sunrise Semester" will be presented six hours a week, Monday through Friday from 6:30 to 7:30 and on Saturday from 7 to 8 A.M. The 1958–59 program, University spokesmen said, also will provide the opportunity for a closer relationship between the student and his instructor and will include optional discussion sessions on the NYU campus and in various suburban communities.

The four courses, the professors who will teach them, and their schedules are:

Classical Civilization H1, a history and literature course on the legacy of Greece and Rome, Dr. Casper J. Kraemer, Jr., professor of archaeology and classics, Monday, Wednesday, and Friday from 6:30 to 7 A.M.

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*Quoted from a News Release, July 16, 1958, Office of Information Services, New York University.

Mathematics N1, an introductory course, Dr. Hollis R. Cooley, professor of mathematics, Monday, Wednesday, and Friday from 7 to 7:30 A.M.

English H5, a "Great Books" course, Dr. David H. Greene, associate professor of English, Tuesday and Thursday from 6:30 to 7 A.M. and Saturday from 7 to 7:30 A.M.

Government S1, a study of the governmental process with particular emphasis on American theory and practice, Dr. Morley Ayearst, associate professor of government, Tuesday and Thursday from 7 to 7:30 A.M. and Saturday from 7:30 to 8 A.M.

Qualified undergraduate applicants can enroll for as many as four courses, earning three credits for each upon satisfactory completion of requirements that include home quizzes, term papers, and final examinations at NYU. Special arrangements will be made for physically-handicapped students. The courses, part of the basic requirements of the Program of Coordinated Liberal Studies of NYU's Washington Square College of Arts and Sciences, will be offered to television viewers under the supervision of Dr. Thomas Clark Pollock, dean of the College.

In addition to the home-television lectures, NYU will offer periodic discussion sessions at its Washington Square Center for each of the courses. Where registration warrants it, there also will be discussion sessions in suburban communities in Westchester County, New Jersey, Long Island, and Connecticut. The discussion groups will be led by the course professors or their departmental colleagues.

A student who takes all eight courses offered during the year can earn 24 academic credits toward a bachelor of arts degree. Persons who do not wish to enroll for degree credit, but who wish to take one or more courses and receive recognition for satisfactory completion, may register for certificate credit.

REPORT OF THE COMMISSION ON MATHEMATICS

ROBERT E. K. ROURKE, Executive Director, Commission on Mathematics

The Commission on Mathematics of the College Entrance Examination Board, appointed in 1955 for the purpose of improving the high school curriculum in college preparatory mathematics, published its report in December 1958.

The report is in two parts. The first part sets forth the proposals of the Commission and their reasons for making them; the second part consists of appendices designed to amplify certain of the Commission's recommendations and to provide useful materials for teachers.

To meet contemporary needs, the report presents the following nine-point program for college-capable students:

1. Strong preparation *both* in concepts *and* in skills for college mathematics at the level of calculus and analytic geometry.
2. Understanding of the nature and role of deductive reasoning—in algebra as well as in geometry.
3. Appreciation of mathematical structure ("patterns")—for example, properties of natural, rational, real, and complex numbers.

4. Judicious use of unifying ideas—set, variable, function, and relation.
5. Treatment of inequalities along with equations.
6. Incorporation with plane geometry of some coordinate geometry and also essentials of solid geometry and space perception.
7. Introduction in grade 11 of fundamental trigonometry—centered on coordinates, vectors, and complex numbers.
8. Emphasis in grade 12 on elementary functions (polynomial, exponential, circular).
9. Recommendation of additional alternative units for grade 12: *either* introductory probability with statistical applications *or* an introduction to modern algebra.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1341. *Proposed by A. A. Mullin, University of Illinois*

If $w=z^2$, $z=x+iy$, express $\mathrm{Re}(w)$ and $\mathrm{Im}(w)$ in terms of x and y.

E 1342. *Proposed by D. J. Newman, AVCO Research Division, Wilmington, Mass.*

If x and y are positive, prove that $x^y+y^x>1$.

E 1343. *Proposed by W. F. Cheney, University of Hartford, Connecticut*

Through an arbitrarily-selected point within a plane triangle, a straight line is drawn which bisects the area of the triangle. What is the probability of more than one solution for a given point?

E 1344. *Proposed by A. J. Goldman, National Bureau of Standards, Washington, D. C.*

Let $\{x_n^{(1)}\}$, $\{x_n^{(2)}\}$, $\cdots$, $\{x_n^{(m)}\}$ be m sequences of nonzero real numbers. Prove that there exists an integer i $(1\leq i\leq m)$ and an ascending sequence $\{n_r\}$

of positive integers ($r=1, 2, \cdots$) such that each of the sequences

$$\{x_{n_r}^{(i)}/x_{n_r}^{(1)}\}, \{x_{n_r}^{(i)}/x_{n_r}^{(2)}\}, \cdots, \{x_{n_r}^{(i)}/x_{n_r}^{(m)}\}$$

is convergent.

E 1345. *Proposed by I. J. Schoenberg, University of Pennsylvania*

There are given $2n$ points on the x-axis: $x_1<x_2<\cdots<x_{2n}$, $(n\geqq 1)$. For convenience we also write $x_0=x_1$, $x_{2n}=x_{2n+1}$ and determine, for each $k=1, \cdots, n$, a point ξ_k between x_{2k-1} and x_{2k} satisfying the linear equation

$$(\xi_k - x_{2k-2})(\xi_k - x_{2k-1}) = (\xi_k - x_{2k})(\xi_k - x_{2k-1}),$$

thus obtaining n points $\xi_1<\xi_2<\cdots<\xi_n$. Finally, let

$$\gamma_k = (x_{2k} + x_{2k+1})/2, \qquad k = 0, 1, \cdots, n.$$

Show that the relation

$$\sum_{k=1}^{n} (\gamma_k - \gamma_{k-1}) f(\xi_k) = \int_{x_1}^{x_{2n}} f(x)dx$$

holds for any continuous function $f(x)$ which is linear in each of the intervals (x_k, x_{k+1}), $k=1, \cdots, 2n-1$.

SOLUTIONS

Covering a Square

E 1311 [1958, 284]. *Proposed by G. K. Wenceslas, Santa Monica, Calif.*

The dissection of a unit square into one $1/8\times 1$ and two $1/2\times 7/8$ rectangular pieces shows that it is possible to cover the unit square with three sets each having diameter $d=\sqrt{65}/8$. Prove that the unit square cannot be covered by three sets all of which have diameter less than d.

Solution by Joe Lipman, University of Toronto. Let the unit square (0, 0), (0, 1), (1, 1), (1, 0) be covered by three sets S_i of diameter $D_i<\sqrt{65}/8$, $i=1, 2, 3$. Since one of the S_i contains two vertices, suppose, without loss of generality, that S_1 contains (0, 0) and (0, 1). Then the region common to the circles of radius D_1 and having these vertices as centers contains S_1 but contains none of the points (1/8, 0), (1/8, 1), (1, 1/2). One of S_2, S_3, say S_2, must contain two of these points, and, D_2 being less than $\sqrt{65}/8$, the two points must be (1/8, 0) and (1/8, 1). The region common to the circles of radius D_2 and having these two points as centers contains S_2, but contains neither of the points (1/4, 0), (1, 1). These must then belong to S_3, but the distance between them is greater than $\sqrt{65}/8$, and so the situation is impossible.

Also solved by L. R. Bragg, Vern Hoggatt, A. R. Hyde, D. C. B. Marsh, C. F. Pinzka, L. A. Ringenberg, and the proposer.

Willie's Doodling

E 1312 [1958, 284]. *Proposed by C. F. Pinzka, University of Cincinnati*

"What kind of doodling is that?" I asked, noticing that Willie had the following steps on his paper:

(0) x_1y_2

(1) $x_3y_4y_5x_6$

(2) $x_7y_8y_9x_{10}y_{11}y_{12}y_{13}y_{14}$

"It's sequential doodling," replied Willie. "I get each new step from the last step by inserting an x after each y, a y after each x, and continuing the subscripts in order."

"Does your doodling have any interesting properties?" I asked in jest.

"Oh, yes," said Willie. "In step n you will find that the sum of the kth powers of the x subscripts equals the sum of the kth powers of the y subscripts for $k=0, 1, \cdots, n$. In fact, you can use any numbers in arithmetic progression for the subscripts and this property will still hold."

Prove that Willie's assertions hold for all (nonnegative integers) n.

Solution by I. C. Gentry, Wake Forest College, N. C. Let the subscripts be any numbers in arithmetic progression and let us employ mathematical induction. The cases $n=0$ and $n=1$ are easily verified. Assume Willie's statement is true for all cases up through the nth and consider the $(n+1)$th sequence

$$x_a y_{a+d} y_{a+2d} x_{a+3d} \cdots y_b x_{b+d} x_{b+2d} y_{b+3d} \cdots,$$

where $b=a+2^{n+1}d$ and the sequence contains 2^{n+2} terms. The induction hypothesis applies for $k=0, 1, \cdots, n$ to each half, and hence to the whole, of this sequence.

Now consider $k=n+1$. Set $D=2^{n+1}d$. Then for every x subscript s in the first half of the sequence there is a y subscript $s+D$ in the second half, and for every y subscript t in the first half there is an x subscript $t+D$ in the second half. Let S_x represent the sum of the $(n+1)$th powers of the x subscripts and let S_y represent the sum of the $(n+1)$th powers of the y subscripts. Then

$$S_x = \sum s^{n+1} + \sum (t+D)^{n+1}, \qquad S_y = \sum t^{n+1} + \sum (s+D)^{n+1}.$$

Expanding the terms in parentheses by the binomial theorem and using the fact that $\sum s^k = \sum t^k$ for $k \leq n$, it now follows that $S_x = S_y$, and Willie's statement holds for the $(n+1)$th case.

Also solved by Peter Beisswanger, Joe Lipman, D. C. B. Marsh, Helen Marston, Benjamin Sapolsky, and the proposer.

Quadratic with Roots Exterior to the Unit Circle

E 1313 [1958, 274]. *Proposed by Roger Pinkham, Princeton University*

If $pz^2-qz+1=0$, then what values of p and q yield roots exterior to the unit circle?

Solution by Howard Eves, University of Maine. The assumption is that p and q are real. The given equation has its roots exterior to the unit circle if and only if the equation $z^2-qz+p=0$ has its roots interior to the unit circle. Denote the roots of this latter equation by α and β. Since $\alpha+\beta=q$, $\alpha\beta=p$, and p and q are real, it follows that α and β are both real or are non-real conjugate complex numbers.

Case 1. (α and β real)

Suppose $|\alpha|<1$, $|\beta|<1$. If α and β are both positive we have

$$(\alpha-1)(\beta-1)=\alpha\beta-(\alpha+\beta)+1>0, \qquad 0<\alpha\beta<1,$$

whence $0<p<1$ and $q<p+1$. If α and β are both negative we have

$$(\alpha+1)(\beta+1)=\alpha\beta+(\alpha+\beta)+1>0, \qquad 0<\alpha\beta<1,$$

whence $0<p<1$ and $-q<p+1$. If α, say, is positive and β negative we have

$$(\alpha-1)(\beta-1)=\alpha\beta-(\alpha+\beta)+1>0, \qquad -1<\alpha\beta<0,$$

whence $-1<p<0$ and $q<p+1$. If one root, say α, is zero, then $\beta=q$ and we have $p=0$, $|q|=|\beta|<1=p+1$. Summarizing the possible situations we see that if $|\alpha|<1$ and $|\beta|<1$, then $0\leqq|p|<1$ and $|q|<p+1$.

By a similar analysis, we can show that if we do not have both $|\alpha|<1$ and $|\beta|<1$, then either $|q|\geqq p+1$ or $|p|\geqq 1$. It follows that if $0\leqq|p|<1$ and $|q|<p+1$, then $|\alpha|<1$ and $|\beta|<1$.

Case 2. (α and β non-real conjugate complex numbers)

Set $\alpha=x+iy$, $\beta=\bar{\alpha}=x-iy$. Then $0<(|x|-1)^2+y^2=x^2+y^2+1-2|x|$, whence $|q|=|\alpha+\beta|=2|x|<x^2+y^2+1=p+1$. Now $|\alpha|=|\beta|<1$ if and only if $p=|\alpha\beta|=|\alpha|^2<1$.

The results established in cases 1 and 2 show that $pz^2-qz+1=0$ has its roots exterior to the unit circle if and only if $0\leqq|p|<1$ and $|q|<p+1$, that is, if and only if in the pq-plane the point (p, q) lies in the interior of the triangle whose vertices are $(1, 2)$, $(-1, 0)$, $(1, -2)$.

Also solved by D. S. Adorno, Merrill Barnebey, A. P. Boblétt, L. R. Bragg, J. L. Brown, Jr., A. G. Clark, E. L. Ellis and D. L. Muench (jointly), José Gallego-Diaz, Michael Goldberg, A. R. Hyde, D. C. B. Marsh, C. S. Ogilvy, J. D. Steben and H. W. Vayo (jointly), R. J. Wagner, Dale Woods, David Zeitlin, and the proposer.

In general these solutions derived various conditions on p and q necessary for the roots of the quadratic to be exterior to the unit circle, but failed to establish conditions which are both necessary and sufficient.

A Summation

E 1314 [1958, 284]. *Proposed by J. M. Gandhi, Jain Engineering College, Gurukul, Panchkoola, India*

Let $S(n, i)=\sum a_1 \cdots a_i$, where the sum is taken over all possible integral choices of the a's such that $1\leqq a_1\leqq a_2\leqq \cdots \leqq a_i\leqq n$, and put $S(0, i)=0$, $S(n, 0)=1$. Prove that for all positive integral n

$$(1) \qquad \sum_{k=0}^{n-1} (-1)^k (n-k)!S(n-k, k) = 1.$$

Solution by E. P. Starke, Rutgers University. Note first, from the formation of $S(n, i)$, the obvious relation

$$(2) \qquad S(n, i) = nS(n, i-1) + S(n-1, i).$$

Now, in (1), replace each term $(n-j)!S(n-j, j)$ by its equal

$$(n-j+1)!S(n-j, j) - (n-j)!(n-j)S(n-j, j)$$

and recombine by taking the second part of each term with the first part of the following, obtaining

$$\begin{aligned}(n+1)!S(n, 0) - n!\{nS(n, 0) + S(n-1, 1)\} + \cdots \\ + (-1)^{j+1}(n-j)!\{(n-j)S(n-j, j) + S(n-j-1, j+1)\} \\ + \cdots + (-1)^{n}1!S(1, n-1) = 1.\end{aligned}$$

But, by (2), this becomes at once

$$\begin{aligned}(n+1)!S(n+1, 0) - n!S(n, 1) + \cdots + (-1)^{j+1}(n-j)!S(n-j, j+1) \\ + \cdots + (-1)^{n}1!S(1, n) = 1.\end{aligned}$$

Thus, since (1) is obvious for $n=1$, we have the proof by induction.

Also solved by E. L. Ellis, Joe Lipman, D. C. B. Marsh, C. F. Pinzka, and Benjamin Sapolsky. Late solution by J. H. Hodges.

A Composite Magic Square

E 1315 [1958, 285]. *Proposed by R. L. Caskey, Oklahoma State University*

Let M represent a magic square of order n, the elements of which are the positive integers from 1 to n^2. Let M_i represent the magic square obtained by replacing each element k of M by $k+(i-1)n^2$. Prove that the square array of elements obtained by replacing each element i of M by M_i is a magic square of order n^2.

Solution by C. F. Pinzka, University of Cincinnati. The row (column, diagonal) sum of M_i is $n(n^2+1)/2+(i-1)n^3 = in^3 - n(n^2-1)/2$. In the final square this sum is $[n(n^2+1)/2]n^3 - n[n(n^2-1)/2] = n^2(n^4+1)/2$ as required. Furthermore, since $k+(i-1)n^2$ represents the consecutive integers from 1 to n^4 as k and i range from 1 to n^2, the resulting array is a magic square of order n^2.

If each element i of a magic square P of order p is replaced by M_i, the same argument shows the resulting array to be a magic square of order np. Kraitchik's *Mathematical Recreations* (1st ed., p. 170) discusses these composite magic squares.

Also solved by Tibor Bakos, Peter Beisswanger, Joe Lipman, D. C. B. Marsh, and D. A. Robinson. Late solution by J. H. Hodges.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4816. *Proposed by M. S. Klamkin, AVCO Research and Development, Wilmington, Mass.*

Solve the integral equation

$$\int_0^\infty t^3\phi(x-t)dt = a\left\{\int_0^\infty t^2\phi(x-t)dt\right\}^b,$$

where a and b are independent of x.

4817. *Proposed by I. S. Gál and Stanley Kaplan, Cornell University*

Determine all entire functions $f(z)$ such that $|f(z)| = 1$ whenever $|z| = 1$.

4818. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let d_i be the sides of a complete quadrilateral, and A_{ij} be the vertex on d_i, d_j. Let t_i be the triangle formed by the sides other than d_i, and (O_i) denote the circumcircle of t_i. Denote the Simson line of a point S_i of (O_i) with respect to t_i by D_i.

Then prove that, if D_i and d_i are parallel for all i, (1) the line S_iO_p passes through the vertex A_{qr} $(i,\ p\neq q,\ r)$, and (2) the points S_i all lie on the Miquel circle (O).

4819. *Proposed by P. T. Bateman, University of Illinois*

Suppose k is a given nonnegative integer. Show that every sufficiently large positive integer is a sum of five squares none of which is less than k^2 and no two of which are equal.

4820. *Proposed by K. Mahler, the University, Manchester, England*

Let p be a prime, K the p-adic field, and I the ring of p-adic integers. Further, let $f(x)$ be a function defined for all $x\in I$ with $f(x)\in K$, such that $f(n)=n!$ $(n=0, 1, 2, \cdots)$. Show that the points of discontinuity lie dense in I.

4821. *Proposed by M. P. Drazin, Trinity College, Cambridge, England*

A square matrix B is called normal if its elements are in the complex field

and if it commutes with its transposed conjugate matrix B^* (*i.e.*, if $BB^*=B^*B$); more generally, call a given $n\times n$ matrix, A, m-normal if A can be imbedded, as leading $n\times n$ submatrix, in some normal $m\times m$ matrix B. Find the greatest value of n for which every (complex) $n\times n$ matrix is $(n+1)$-normal, and show that every square matrix is m-normal for all sufficiently large m.

What are the corresponding results when A, B are restricted to be real?

SOLUTIONS

Vogt's Determinant

4771 [1958, 47]. *Proposed by Leonard Carlitz, Duke University*

Muir (*Contributions to the History of Determinants* 1900–1920) reproduces the assertion of Vogt that the determinant

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2^2 & \cdots & 2^n & 2^{n+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ n & n^2 & \cdots & n^n & n^{n+1} \\ \frac{n}{2} & \frac{n^2}{3} & \cdots & \frac{n^n}{n+1} & \frac{n^{n+1}}{n+2} \end{vmatrix}$$

vanishes for $n\geqq 2$ and even. Prove the truth of this assertion. Also show that $D_n\neq 0$ for n odd.

Solution by Gabor Szegö, Stanford University. We show that the associated linear equations

$$x_0\nu + x_1\nu^2 + \cdots + x_n\nu^{n+1} = 0, \qquad 1 \leqq \nu \leqq n,$$

$$x_0\frac{n}{2} + x_1\frac{n^2}{3} + \cdots + x_n\frac{n^{n+1}}{n+2} = 0,$$

have nontrivial solutions for $n\geqq 2$ and even, and only the trivial solution for n odd.

Writing $x_0z+x_1z^2+\cdots+x_nz^{n+1}=f(z)$, these equations read:

$$f(\nu) = 0,\ 1 \leqq \nu \leqq n; \qquad \int_0^n f(z)dz = 0,$$

so that $f(z)=x_nz(z-1)\cdots(z-n)$. Thus the assertion is equivalent to the fact that

$$I_n = \int_0^n \binom{z}{n+1} dz$$

is 0 for $n\geqq 2$ and even, and $I_n\neq 0$ for n odd.

Now

$$\binom{z}{n+1} = (-1)^{n+1}\binom{n-z}{n+1},$$

so that $I_n = -I_n$ if n is even.

If n is odd, we use

$$\binom{z}{n+1} + \binom{z}{n+2} = \binom{z+1}{n+2},$$

so that

$$I_n = \int_0^n \binom{z+1}{n+2}dz - \int_0^n \binom{z}{n+2}dz = \left(\int_1^{n+1} - \int_0^n\right)\binom{z}{n+2}dz$$

$$= -\int_0^1 \binom{z}{n+2}dz + \int_n^{n+1}\binom{z}{n+2}dz = -2\int_0^1 \binom{z}{n+2}dz,$$

in view of

$$\binom{z}{n+2} = -\binom{n+1-z}{n+2}.$$

Thus I_n is negative.

Also solved by A. C. Aitken, Robert Breusch, D. R. Brillinger, A. G. Konheim and J. A. Navarro, Joe Lipman, F. D. Parker, F. W. Ponting, Suja Ram, W. E. Roth, E. M. Wright, J. Van Yzeren, and the proposer.

Consecutive Prime Triplets

4772 [1958, 47]. *Proposed by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.*

It is easy to show that there exist consecutive prime pairs such that their difference is arbitrarily large. Do there exist consecutive prime triplets P_1, P_2, P_3 such that min (P_2-P_1, P_3-P_2) is arbitrarily large?

Solution by P. T. Bateman, University of Illinois. The question of the problem was answered affirmatively by Sierpiński [*Colloquium Math.* vol. 1, 1948, pp. 193–194]. The following stronger results have been since obtained. Erdös [*Publicationes Math. Debrecen*, vol. 1, 1949, pp. 33–37] proved that for any positive number C there exist consecutive prime triplets such that $\min(P_2-P_1, P_3-P_2) > C \log P_3$. Walfisz [*Doklady Akad. Nauk SSSR* (N. S.) vol. 90, 1953, pp. 711–713] proved that for almost all primes p the distance to the closest prime on either side is greater than $(\log p)/(\log\log\log p)^2$. Prachar [*Monatsh. Math.* vol. 58, 1954, pp. 114–116] showed that Walfisz's result is still true if $(\log\log\log p)^2$ is replaced by any function of p which tends to infinity with p.

The following proof is similar to Sierpiński's but differs somewhat in detail.

Let q be any prime greater than 2. Then $(q-1)!-1$ and $q!$ are relatively prime. In fact, the prime factors of $q!$ are the primes not exceeding q. Clearly $(q-1)!-1$ is not divisible by any of the primes less than q, while $(q-1)!-1 \equiv -2 \pmod{q}$ by Wilson's theorem. Since $(q-1)!-1$ and $q!$ are relatively prime, Dirichlet's theorem guarantees the existence of a prime p such that

$$p \equiv (q-1)! - 1 \pmod{q!}.$$

Now the q integers following p and the $q-2$ integers preceding p are composite, since

$$p \pm k + 1 \equiv p + 1 \equiv (q-1)! \equiv 0 \pmod{k}$$

if $2 \leqq k \leqq q-1$, and since $p+2 \equiv (q-1)!+1 \equiv 0 \pmod{q}$. But q may be taken as large as desired and so the assertion of Sierpiński is established.

Also solved by Robert Breusch, W. E. Briggs and D. Hawkins, N. J. Fine and Emil Grosswald, John B. Kelly, Leo Moser, and Suja Ram.

A Class of Irrational Numbers

4773 [1958, 125]. *Proposed by Paul Erdös.*

Let $n_1 \leqq n_2 \leqq \cdots$ be a sequence of positive integers. Assume $n_k^{1/2^k} \to \infty$. Show that $\sum_{k=1}^{\infty} 1/n_k$ is irrational. (It is easy to construct $\sum 1/n_k$ rational with $n_k > A^{2^k}$ for every fixed A.)

Solution by Rimhak Ree, University of British Columbia. If $\sum_{k=1}^{\infty} 1/n_k = b/a$, where a and b are relatively prime positive integers, then we would have

$$n_1 n_2 \cdots n_{k-1} \sum_{t=0}^{\infty} 1/n_{k+t} \geqq 1/a \tag{1}$$

for all $k = 1, 2, \cdots$. We shall derive a contradiction.

Let $n_k^{1/2^k} = \alpha_k$. Then $\alpha_k \to \infty$ implies the existence of an integer $r > 0$ such that $\alpha_{r+t} > a+1$ for $t = 0, 1, 2, \cdots$ and $\alpha_i \leqq \alpha_r$ for $i = 1, 2, \cdots, r-1$. We have

$$\begin{aligned} n_1 n_2 \cdots n_{r-1}/n_{r+t} &\leqq \alpha_r^{2+2^2+\cdots 2^{r-1}}/n_{r+t} \\ &\leqq \alpha_{r+t}^{2^t(2^r-2)}/\alpha_{r+t}^{2^{r+t}} = \alpha_{r+t}^{-2^{t+1}} < (a+1)^{-(t+1)}, \end{aligned}$$

since $n_r \leqq n_{r+t}$ implies $\alpha_r \leqq \alpha_{r+t}^{2^t}$. Therefore we have

$$n_1 n_2 \cdots n_{r-1} \sum_{t=0}^{\infty} 1/n_{r+t} < \sum_{t=0}^{\infty} (a+1)^{-(t+1)} = 1/a,$$

which contradicts (1).

Also solved by Robert Breusch.

Two Summations

4775 [1958, 125]. *Proposed by Leonard Carlitz, Duke University*

Prove the formulas

$$(1) \qquad \sum_{r=0}^{m} \frac{A_r A_{m-r} A_{n-r}}{A_{m+n-r}} \frac{2m+2n-4r+1}{2m+2n-2r+1} = 1$$

$$(2) \qquad \sum_{r=0}^{m} (-2)^r \frac{A_r A_{m-r} A_{n-r} A_{k-r}}{A_{m+n-r}} \frac{4k-4r+1}{4k-2r+1} = A_{m/2} A_{n/2},$$

where $m \leqq n$, $A_r = 1 \cdot 3 \cdot 5 \cdots (2r-1)/r!$, $A_0 = 1$, $A_x = 0$ for x not an integer, and, in (2), $m+n=2k$.

Solution by A. E. Danese, Mount Morris, N. Y. The proposed formulas are special cases of a familiar result due to Adams (Whittaker and Watson, *A Course of Modern Analysis*, p. 331, 11):

$$P_m(z)P_n(z) = \sum_{r=0}^{m} \frac{A_{m-r} A_r A_{n-r}}{A_{n+m-r}} \frac{2n+2m-4r+1}{2n+2m-2r+1} P_{n+m-2r}(z)$$

where $P_n(z)$ is the Legendre polynomial of order n, m and n are positive integers with $m \leqq n$, and $A_r = 1 \cdot 3 \cdots (2r-1)/r!$

Now using $P_n(1) = 1$, $P_{2n}(0) = (-2)^{-n} A_n$, $P_{2n+1}(0) = 0$, for all positive integers n (*ibid.*, p. 303, ex. 1), the results are immediate.

Also solved by P. Henrici, B. S. Popov, G. Szegö, Chih-yi Wang, and the proposer.

A Summation Representing an Exponential Function

4776 [1958, 451]. *Proposed by D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

If $|\alpha| < 1/e$, then

$$\sum_{n=0}^{\infty} \frac{(z+\alpha n)^n}{n!}$$

represents an entire function. Prove in fact that it is a simple exponential, $Ae^{\lambda z}$.

Solution by Leonard Carlitz, Duke University. It is known (see, e.g., Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, p. 302, no. 214) that

$$\sum_{n=0}^{\infty} \frac{(n+\beta)^n \alpha^n}{n!} = \frac{e^{\beta\mu}}{1-\mu}, \qquad (|\alpha| < 1/e),$$

where $\mu e^{-\mu} = \alpha$. With $z = \alpha\beta$ and $\mu = \log \lambda$, this becomes immediately

$$\sum_{n=0}^{\infty} \frac{(z+\alpha n)^n}{n!} = \frac{e^{z\alpha^\beta}}{1-\mu} = \frac{e^{z\lambda}}{1-\log\lambda}.$$

Also solved by Robert Breusch, James Clunie, N. J. Fine, M. Ganeshavijaranayengar, Emil

Grosswald, A. H. M. Levelt, Immanuel Marx, and the proposer. Late solutions by Norman Greenspan and Robert Weinstock.

The original form of the problem [1958, 125] was also solved by P. T. Bateman, D. R. Brillinger, C. N. Campopiano, Michael Goldberg, J. W. Haake, J. D. E. Konhauser, R. L. London, D. C. B. Marsh, Burton Randol, and Chih-yi Wang.

Sums of Powers of Binomial Coefficients

4777 [1958, 125]. *Proposed by R. P. Pakshirajan, Indian Statistical Institute, Calcutta*

Prove that

$$\lim_{n\to\infty} \frac{n}{2^{3n}}\left\{\binom{n}{0}^3 + \binom{n}{1}^3 + \cdots + \binom{n}{n}^3\right\} = \frac{2}{\pi\sqrt{3}}.$$

Editorial Note. As pointed out by several readers, this is the case $k=3$ of II, 40, pp. 42, 201 in Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. I.

Klamkin points out that it is also the case $f(x)=x^p$ of II, 190, pp. 76, 242 in the same work, vol. I. Wang refers to a formula involving $\binom{n}{0}^3-\binom{n}{1}^3+-\cdots\pm\binom{n}{n}^3$ by A. C. Dixon, *Messenger of Mathematics*, vol. 20 (1891), pp. 79–80.

Other solutions by Robert Breusch, Yoshio Matsuoka, M. R. Mickey, G. W. Petrie III, Chih-yi Wang, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

College Algebra (6th ed.). By Henry L. Rietz, Arthur R. Crathorne, and J. William Peters. Holt, New York, 1958. xiv+385 pp. $4.50.

This text differs from the fifth edition (reviewed this MONTHLY, 1952, vol. 59, pp. 192–194) in about one-fourth of the exercise sets and in a couple of the sections. Representative of the difference is the increase in the price of coffee by twenty-five cents. The changes do little more than require that class work be restricted to one of the editions, unless the course covers only the last half of the book. For, starting with Chapter 11, the only differences seem to be on pages 254, 264, 274 and 275.

JAMES H. MCKAY
Seattle University

Differential Equations: Geometric Theory. By Solomon Lefschetz. Interscience, New York and London, 1957. vi+364 pp. $9.50.

The twelve chapters of this book can be divided into four parts. The first part would consist of the first four chapters, which contain a review and develop-

ment of the material on topology, vectors, matrices and functions of several variables necessary to the rest of the book. This section has chapters on stability and linear systems. In the second part, consisting of four chapters, three chapters are devoted to point stability of systems and the treatment of Poincare, Liapunov and Perron. The last chapter of this part deals with the stability of general periodic systems. In the third part, of two chapters, attention is turned to the results of Poincare and Bendixson on two dimensional systems. The results of Andronov, Pontrjagin and De Baggis on structural stability are discussed. The last section, of two chapters, discusses equations of the second order and the contributions of Cartwright, Littlewood, Levinson and Smith. This covers the self-oscillations of van der Pol, the method of perturbation and applications to Mathieu's equation. There is an appendix on matrices and on the index of a curve. The book is in the form of a monograph in that there are no applications or exercises. However, nineteen problems of a serious nature are listed at the end for the reader's further investigation.

This work forms an important and welcome addition to the literature. Previously part of the material was available in the Annals of Mathematics Studies, in *Nonlinear Vibrations in Mechanical and Electrical Systems*, by H. H. Stoker, and in the translation by Lefschetz of Andronov and Chaikin's *Theory of Oscillations*. Much of the material was not in book form or available to the non-Russian speaking reader. The typography is excellent and many difficult diagrams have been clearly reproduced.

The selection of the material reflects the individuality and wide interests of the author. This is a volume by one of the most prominent workers in the subject and is a volume which anyone interested in the theory of nonlinear ordinary differential equations will want to own.

G. M. PETERSEN
University of New Mexico

Paper Folding for the Mathematics Class. By Donovan A. Johnson. National Council of Teachers of Mathematics, Washington, D. C., 1958, 32 pp. $0.75.

This interesting little booklet gives a fresh viewpoint to geometry. Starting with basic assumptions such as "paper can be folded so that the crease passes through one or two given points," most of the constructions of plane Euclidean geometry can be performed by folding and creasing. They range from simple constructions such as "a line perpendicular to a given straight line," through circle relationships, and even such products as $(x+y)(x-y)=x^2-y^2$.

Some of the later chapters like "Polygons Constructed by Tying Paper Knots," and "Conic Sections," are ingenious but rather tricky to follow. The illustrations are numerous and excellent. It should appeal to students in all levels of high school and beginning college and, of course, to their teachers. Such a fresh outlook on a routine subject is very welcome.

JOSEPHINE P. ANDREE
University of Oklahoma

Advanced Calculus. By Louis Brand. Wiley, New York, 1955. 574 pp. $8.50.

This book, as the author points out in his preface, serves as an excellent introduction to real variable theory. Following an introductory chapter on the development of the number system from an axiomatic standpoint, the author begins to develop his function theory and the theory of limits in a chapter on sequences and series. It is refreshing to find a text at this level which includes several tests for convergence of series in addition to the more familiar ratio, root and comparison tests, although the author properly postpones until later the discussion of the integral test.

With this introduction, the author proceeds to chapters on functions of a real variable, functions of several variables, vectors, the Riemann integral, improper integrals, line integrals, and multiple integrals. Here follows a complete chapter on uniform convergence and its implications in the reversal of order in limiting processes. This chapter includes a discussion of gamma and beta functions. The penultimate chapter in the text proper concerns functions of a complex variable, including the use of contour integration for the evaluation of definite integrals. The last chapter is an introduction to Fourier series, including a brief discussion of the Gibbs phemonenon. The annexation of four appendices, on cluster points, difference equations, difference calculus, and dimensional checks, subjects not usually found in such books and not apropos to any particular chapter, is well worth while and quite commendable.

There are many excellent characteristics of this book. The proofs are clearly written. In many cases, introduction of special devices simplifies proofs and clarifies theorems which more standard procedures may leave obscure. For example, Euler's theorem on homogeneous functions and its converse (page 161), and the proof of uniform continuity of a continuous function over a closed interval and related theorems are proved without recourse to a Heine-Borel theorem (pages 91 ff.). The summarization of the content of each chapter at its close is excellent, and should serve as an aid in teaching from this book as a text.

Despite the high caliber of the presentation, it is marred at times by unfortunate errors. Some of these may be typographical, although their repetition in relatively close juxtaposition might lead one to believe they appeared in the manuscript. One particular blunder which stands out appears on the bottom of page 121, where the author uses $\lim_{x\to 0} x^{1/2}/\log x$ as an indeterminate form to illustrate a portion of his theory. Again, there is a repetition of the error of omitting the radical in the integral for $\sin^{-1} b - \sin^{-1} a$ on page 265; and on pages 316–18, the differential dx is omitted in several integrals.

With the correction of errors, typographical and otherwise, this book will be an excellent text for an introduction to mathematics at the graduate level. There are an ample number of exercises, theoretical and numerical, with answers.

ROBERT E. LOWNEY
Montana State College

Plane Geometry for Colleges. By L. J. Adams. Holt, New York, 1958. v+213 pp. $3.50.

This book is designed for a one-semester, three-hour course in plane geometry for college freshmen. An effort has been made to write a more mature book than that found in some available texts in this field. Objectional statements of some geometry texts, such as "axioms are self-evident truths" and the formulation of definitions for undefined objects are conspicuous by their absence here. Although many plane geometries cover far more theorems and corollaries than this book, it is evident that the author has stressed quality rather than quantity.

The development of material is along essentially three lines: (i) nature of geometry as a logical system, (ii) constructions and mensuration, (iii) meaning and content of theorems and their corollaries. A last chapter points out applications of geometry.

Perhaps, for college students, more on rigor in geometric proofs might have been done, and, although vectors are introduced about one-fourth of the way through this book, they seem to be placed more in the category of a curiosity rather than put to any real geometric use.

These criticisms seem rather finicky when compared with the book's good points. The book appears teachable, and merits consideration as a text for a first course in plane geometry at the college level.

BEN T. GOLDBECK, JR.
Texas Christian University

Introduction to Statistical Reasoning. By Philip J. McCarthy. McGraw-Hill, New York, 1957. xiv+402 pp. $5.75.

Dr. McCarthy's purpose is to present to lower division majors in social sciences, with only high school mathematics, the "elementary methods common to all statistical investigations, regardless of the field" (p. 2).

He accomplishes this aim satisfactorily, and without undue originality, but so have many recent texts such as Sprowls, Croxton and Crowden, and Mills.

His introductory remarks take up the preface and the first two chapters—more space than is necessary. For instance, on page 11 he devotes an entire paragraph to an elaboration of the variability of an individual's responses at various times, an obvious truth requiring merely a sentence.

Chapter 3 presents effectively the rectangular, normal, Poisson, and some irregular density functions, although, oddly enough, he does not name them.

Chapters 4 and 5 discuss location and scale parameters and their estimates. He uses $N-1$ as the denominator of the variance estimate with only a footnote of explanation: " . . . simplifies . . . expressions . . . used in more advanced statistical work" (p. 109). However, elementary students find that dividing the sum of N squared differences from the mean by $N-1$ is a confusing procedure. They are entitled to a fuller explanation of the concept, "bias."

Chapter 6 on random sampling is excellent but not new. Chapter 7 covers

elementary probability very well, using some interesting examples.

Chapter 8 on the binomial distribution demonstrates how that function arises, its meaning and employment in statistical inference, and the precise significance of "95 per cent confidence." Chapter 9 is good on large-sample techniques, as is Chapter 10 on systematic, stratified, and random sampling.

The two final chapters cover chi-square tests and linear regression. His symbol y_{x_i} for "estimated value of the dependent parameter" is unconventional as well as awkward. The usual $\hat{y}_i$ is preferred. He discusses chi-square exclusively in connection with an "$m \times n$" contingency table, relegating to a footnote (p. 310) any other use of that versatile statistic.

A bad omission is the reduction of all discussion of Student's t-distribution to two footnotes, pages 265, 355. Few will agree that this important idea "can best be originated in a second course" (p. ix).

EUGENE H. LEHMAN, JR.
San Diego College for Women

Dynamic Programming. By Richard Bellman. Princeton University Press, Princeton, 1957. xxv+342 pp. $6.75.

This book brings under one cover the introduction and development of the theory of dynamic programming, which to a great extent has appeared previously in many papers scattered throughout many journals and pamphlets.

The class of problems which gave impetus to the development of the dynamic programming approach are multi-stage decision processes as they appear in economics, game theory, logistics, *etc.*

To give a greatly over-simplified idea of the subject, suffice it to say that "n" successive one-dimensional decisions are substituted for one "n" dimensional decision.

The author's aim of providing "an introduction to the mathematical theory" certainly has been accomplished. However, his suggestion to use the book as the text for "a course on the advanced calculus level" seems to be an application of the author's "principle of wishful thinking," since throughout the book there is a disturbing disconnectedness. Due to the scope of the subject covered this may have been unavoidable.

There are an abundant number of exercises at the end of each chapter. These, as well as the examples in the text, cover the wide range of the applicability of dynamic programming. Since the answers to most problems are not given, the designation of the problem sections as "Exercises and Research Problems" should prove rather confusing to the reader as to which is exercise and which is research problem.

Topics covered include: Chapter I: A discussion of a multi-stage allocation process, including existence and uniqueness theorems; the properties of the solution. Chapter II: A similar discussion of a stochastic multi-stage decision process. Chapter III: A more general discussion of dynamic programming proc-

esses; mathematical formulation of these. Chapter IV: Existence and uniqueness theorems, convergence, and stability. Chapter V: Formulation and discussion of the optimal inventory equation and its ramification. Chapters VI and VII: Bottleneck problems, the dual problem, and discussion of several examples. Chapter VIII: A continuous stochastic decision process; several approaches to its solution. Chapter IX: Dynamic programming approaches to several classical and other calculus-of-variation problems. Chapter X: Multi-stage games, including some discussion of nonzero-sum games. Chapter XI: Markovian decision processes.

This book will certainly prove extremely useful to the mathematicians, economists, statisticians, engineers, and operation analysts alike. The mathematician especially will look forward to the "contemplated second volume on a higher mathematical level" in which the author promises to rectify some omissions of the present volume.

ALBERT NEWHOUSE
University of Houston

BRIEF MENTION

Analysis of Research in the Teaching of Mathematics. By Kenneth E. Brown, Specialist for Mathematics, U. S. Department of Health, Education, and Welfare. No. 4. 73 pp. $0.25.

Dr. Brown has done a masterful job summarizing the recent educational research done in the teaching of mathematics. Perhaps one of the most interesting features in this brief booklet is the inclusion of seventy-one questions which are still unanswered and which, in the opinion of the research workers, would be significant topics for further research. Let us hope that each university having a college of education will forward this list to the proper persons.

Man and Number. By Donald Smeltzer. Emerson Books, New York, 1958. viii +114 pp. $2.50.

Perhaps the table of contents is as indicative as any review could be: Chapter One, Man Learns to Count; Two, Number Recording; Three, Early Calculating Devices; Four, The Modern Number System. One should not conclude that Chapter Four contains a rational development of the real numbers or anything as modern as this.

Plane Trigonometry. By John J. Corliss and Winifred V. Berglund, Second Ed. Houghton Mifflin, Boston, 1958. xii+397 pp. $4.00.

This is a book from which it would be quite possible to give a reasonably modern course in trigonometry. It would also be possible to give an old-fashioned course with the emphasis mainly on triangle solving. The study of the graphs of the trigonometric functions before the solution of the right triangle is interesting.

Energy for Man. Windmills to Nuclear Power. By Hans Thirring. Indiana University Press, 1958. 409 pp. $6.95.

Operational Mathematics. By Ruel V. Churchill. Second Ed. McGraw-Hill, New York, 1958. ix+337 pp. $7.00.

The revised second edition of this book on operational mathematics and integral transformations is a welcome addition to the growing literature on advanced mathematics. An advanced calculus background is assumed along with work on partial differential equations. A number of the proofs have been replaced with better versions, and several chapters have apparently been entirely rewritten.

Switching Circuits and Logical Design. By Samuel H. Caldwell. Wiley, New York, 1958. xvii+686 pp. $14.00.

The Russian Literature of Satellites, Parts I and II. International Physical Index, New York, 1958. Part I, $10.00, Part II, $12.50.

The translation into English of a series of Russian papers on the subject of satellites is indeed timely and welcome.

A Comprehensive Bibliography on Operations Research through 1956 with supplement for 1957. Wiley, New York, 1958. xi+188 pp. $6.50.

Finite Queuing Tables. By L. G. Peck and R. N. Hazelwood. Operations Research and Mathematics Group, Cambridge, Mass. Wiley, New York, 1958. xvi+210 pp. $8.50.

Extensive, legible tables of the queuing function. This book was produced by photo-offset from machine printing from Univac One, thus requiring no human editing of the numerical portions.

Physics Express, Automation Express, Electronics Express. International Physical Index, Inc., New York. Any one Express $57.50 a year; any two Expresses $100 a year; all three Expresses $150 a year. (Free samples on request.)

Three new periodicals each appearing ten times a year with comprehensive digests of current Russian periodical literature based on the contents of 73 Russian journals

Arithmetic for Colleges. By Harold D. Larsen. Macmillan, New York, 1958. xiii+286 pp. $5.50.

Professor Larsen has rewritten the exercises in his earlier, 1950, text designed primarily for "elementary school teachers who have had no contact with arithmetic since their own elementary school days." It is sad to report that there are many elementary school teachers in this situation, but facts must be faced even when they are unpleasant. Professor Larsen's book has already helped alleviate the situation.

Figurets. By J. A. H. Hunter. Oxford University Press, New York, 1958. x+116 pp. $3.50.

The author of *Fun with Figures* brings us another collection of mathematical and near-mathematical puzzles—some old, some new.

Mathematical Excursions. By Helen A. Merrill. Dover, New York, 1958. 145 pp. $1.00.

This reprint of Professor Merrill's earlier, 1933, book is apparently unchanged from the original edition. It contains 90 interesting puzzles woven into a fabric of mathematical excursions. Your bookstore should carry this on its shelf of paperbacks.

An Introduction to the Dynamics of Airplanes. By H. Norman Abramson. Ronald Press, New York, 1958. viii+225 pp. $4.50.

An undergraduate text in aeronautical engineering which includes the use of matric applications to vibration problems.

Applications of Tensor Analysis. By A. J. McConnell. Dover, New York, 1958. 318 pp. $1.85.

This is a reprint, with a more apt title, of the author's *Applications of the Absolute Differential Calculus* published in 1931.

The Theory of Functions of a Real Variable. Vols. I and II. By E. W. Hobson. Dover, New York, 1958. xv+736 pp. in Vol. I and x+780 pp. in Vol. II. $6.00.

Thanks to Dover Publications for making available an inexpensive reprint of Hobson's well-known work.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Lloyd J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

NATIONAL ACADEMY OF SCIENCES—NATIONAL RESEARCH COUNCIL DIVISION OF MATHEMATICS FELLOWSHIP AND RESEARCH OPPORTUNITIES

The Division of Mathematics calls attention to the fact that several foundations and offices offer financial support for research in mathematics during the year 1959–60.

A number of fellowships will be made available, as well as opportunities for mathematicians to engage in basic research. A partial list, with comments, is given below.

1. *National Science Foundation.* The National Science Foundation sponsors various fellowship programs in the sciences, including mathematics.

Predoctoral fellowships are awarded annually at the first-year, intermediate, and terminal-year levels of graduate study. Applications for 1959–60 will be available in October 1958 from the National Academy of Sciences—National Research Council until the closing date in early January 1959; Award date—March 16, 1959.

Science Faculty fellowships for college science teachers (including mathematics), who plan to continue teaching and who wish to increase their competence as teachers, are offered semiannually at the present time. Eligibility requirements include a baccalaureate degree and three years of full-time experience in teaching natural science subjects at the collegiate level. Awarded annually. The program will be open from May to October. Awards will be announced in early December. Address all inquiries for information and applications to National Science Foundation, Division of Scientific Personnel and Education, Washington 25, D. C.

Postdoctoral fellowships (in making inquiry about postdoctoral awards specify program).

(1) *Regular* postdoctoral fellowships—primarily for recipients of the doctoral degree; awarded semiannually. Program for 1959–60 concurrent with predoctoral program (see above) except that program closes in December. Information and applications will be available from NAS-NRC. The program will also be open from July to early September 1959. Awards are announced in March and October.

(2) *Senior* postdoctoral fellowships—are open to persons who have held a doctoral degree in one of the basic fields of science for a minimum of five years at time of application, or who have had equivalent training and experience. Awarded annually. Applications are available from the National Science Foundation, Division of Scientific Personnel and Education, Washington 25, D. C. The program will be open from May to October. Awards will be announced in early December.

Research Grants. The National Science Foundation also supports basic research in the mathematical sciences by means of grants. While proposals for such support are accepted at any time, individuals desiring support to begin in the summer or at the beginning of a fall semester should submit their proposals in the mathematical sciences preferably by November 1; persons desiring support to begin in the spring semester should submit their proposals preferably by May 1. Instructions for the preparation of proposals, contained in a booklet entitled Grants for Scientific Research, may be obtained upon request from the Program Director for Mathematical Sciences, National Science Foundation, Washington 25, D. C.

2. *Office of Naval Research.* The Office of Naval Research, through contracts with universities and other organizations, supports basic research in broadly-selected fields of mathematics. Proposals should be directed to the Mathematics Branch, Office of Naval Research, Washington 25, D. C. In addition, postdoctoral research associateships in pure mathematics are being established under contracts with the ONR at selected universities. For details and application forms write to the above address.

3. *Air Force Office of Scientific Research.* The Air Force Office of Scientific Research supports research in mathematics directly through contracts with colleges, universities, foundations, and industrial laboratories. Such organizations are encouraged to submit proposals for research in mathematical fields in which they specialize. Proposals should be mailed to the Commander, Air Force Office of Scientific Research, Attn: Mathematics Division, Washington 25, D. C.

4. *Office of Ordnance Research, U. S. Army.* Among the functions of the Office of Ordnance Research is the support of basic research in mathematics. Proposals for projects are ordinarily made by individual scientists or groups of scientists in a form which leads to a contract between the Office of Ordnance Research and a university or research laboratory. For further information write to Commanding Officer, Office of Ordnance Research, Box CM, Duke Station, Durham, North Carolina.

5. *Fulbright Awards—Public Law 584 (79th Congress).* Approximately 400 awards are offered annually for university lecturing and postdoctoral research in all academic fields in Argentina, Australia, Brazil, Burma, Chile, Colombia, Ecuador, India, New Zealand, Pakistan, Paraguay, Peru, the Philippines, and Thailand (competition for the preceding countries closes April 15, 1959); Austria, Belgium-Luxembourg, Republic of China, Denmark, Finland, France, Germany, Greece, Iceland, Iran, Ireland, Israel, Italy, Japan, the Netherlands, Norway, Turkey, and the United Kingdom, including colonial dependencies (competition for the latter countries closes October 1, 1959). In both cases awards are for the academic year 1960–61 (the 1959–60 competition for Europe closes October 1, 1958), but in the former group of countries the academic year begins in the spring or summer instead of the autumn. Awards are payable in foreign currency and usually include travel for the grantee, but not for members of his family, and a maintenance allowance, which may be adjusted in relation to the number of accompanying dependents up to four. Requests for information should be addressed to the Committee on International Exchange of Persons, Conference Board of Associated Research Councils, 2101 Constitution Avenue, Washington 25, D. C.

6. *National Bureau of Standards. Naval Research Laboratory. Air Research and Development Command.* Postdoctoral resident research associateships are available in a variety of sciences including mathematics and are tenable at the Washington, D. C. and Boulder, Colorado laboratories of the National Bureau of Standards; at the Naval Research Laboratory in Washington, D. C.; and at selected development and research centers of the Air Research and Development Command. Necessary facilities and equipment incident to the research of the associate will be provided. For further information write to Fellowship Office, National Academy of Sciences—National Research Council, 2101 Constitution Avenue, Washington 25, D. C. Applications for the 1959–60 program must be filed on or before January 19, 1959.

7. *Atomic Energy Commission.* The Division of Research of the Atomic Energy Commission, through contracts with universities and other organizations, supports research in the fields of numerical analysis, digital-computer design, programming research, and related topics. Proposals should be submitted to the Division of Research, Atomic Energy Commission, Washington 25, D. C.

Brookhaven National Laboratory. Brookhaven National Laboratory, operated by Associated Universities, Inc., under contract with the Atomic Energy Commission, offers postdoctoral research appointments in mathematics. Appointments are for one year, and may be renewed for one additional year. U. S. citizenship is not required, although Atomic Energy Commission approval is a prerequisite. The appointee may work in numerical analysis, digital computing, mathematical physics, differential equations, probability and statistics, and various specialized branches, including reactor theory, hydrodynamics, and orbit theory. Computational facilities are available. Letters from candidates should give details of personal history, scientific background, and qualifications; two letters of recommendation, one from the applicant's research professor, are required. Applications should be directed to M. E. Rose, Head, Applied Science Division, Brookhaven National Laboratory, Upton, Long Island, New York.

MAGAZINES FOR FRIENDSHIP, INC.

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PERSONAL ITEMS

Professor Jewell H. Bushey of Hunter College represented the Association at the inauguration of Richard Heathcote Heindel as President of Wagner Lutheran College, and the observance of the 75th Anniversary of the Founding of the College on the afternoon of Sunday, October 19, 1958.

Professor H. D. Brunk, University of Missouri, will be a Fulbright lecturer at the University of Copenhagen, Copenhagen, Denmark, from July 1958 until May 1959.

Miss Dorothy I. Koehler, University of Kentucky, is a Fulbright Award teacher at Cornwall College, Montego Bay, Jamaica, British West Indies.

University of California, Berkeley: Professor G. P. Hochschild, University of Illinois, and Associate Professor M. A. Rosenlicht, Northwestern University, have been appointed to professorships; Assistant Professor H. J. Bremermann, University of Washington, and Associate Professors S. R. Fary, University of Montreal, and R. J. De Vogelaere, University of Notre Dame, have been appointed to associate professorships; Assistant Professors H. O. Cordes, University of Southern California, R. L. Vaught, University of Washington, J. W. Woll, Jr., Lehigh University, and Dr. J. F. Treves of Paris, have been appointed to assistant professorships; Dr. Charles Ballantine, Dr. C. W. Clark, University of Washington, Dr. Denis Rutovitz and Dr. Roy Takenaga have been appointed to instructorships; Professor Andrzej Mostowski, University of Warsaw, has been appointed Visiting Professor, 1958–59; Professor Emeritus George Pólya, Stanford University has been appointed Visiting Professor for the Spring term 1959; Professor J. C. Shepherdson, University of Bristol, England, will be Visiting Associate Professor, 1958–59; Dr. S. R. Foguel, New York University, Dr. Avner Friedman, Indiana University, Dr. Helmut Klingen, University of Gottingen, Germany, Dr. Anil Nerode, University of Chicago, and Assistant Professor C. J. Titus, University of Michigan, will be Visiting Assistant Professors during 1958–59; Assistant Professor J. R. Shoenfield, Duke University will be Visiting Assistant Professor in the Spring term 1959; Dr. E. C. Zeeman, Cambridge University, England, is Visiting Assistant Professor for the Fall term 1958; Associate Professors L. A. Henkin, H. D. Huskey, M. H. Protter, and Abraham Seidenberg, have been promoted to professorships; Assistant Professors Harley Flanders and Henry Helson have been promoted to associate professorships; Assistant Professor W. G. Bade has been awarded a Senior Postdoctoral National Science Foundation Fellowship for study at Yale, 1958–59; Dr. Leo Breiman, who was a Visiting Assistant Professor 1957–58, has been awarded a Postdoctoral National Science Foundation Fellowship for study in Paris, 1958–59; Assistant Professor James Eells, Jr., will be Visiting Assistant Professor at Columbia University 1958–59; Professor M. H. Protter has accepted an invitation to spend the Spring term 1959 at the Institute for Advanced Study, Princeton; Assistant Professor Emery Thomas has been awarded a Postdoctoral National Science Foundation Fellowship for study at Oxford, England, 1958–59.

The University of Connecticut: Associate Professor J. C. Montgomery has been appointed Acting Head of the Department for the academic year 1958–59; Professor C. H. W. Sedgewick has retired, and has accepted a position as Statistician at the Bureau of the Census, Washington, D. C.; Assistant Professor R. P. Gosselin has been promoted

to Associate Professor; Dr. Geraldine A. Coon, Taylor Instrument Companies, Rochester, New York, and Assistant Professor Chester Feldman, University of New Hampshire, have been appointed Assistant Professors; Mrs. Grazyna Gross has been appointed Instructor at the Stamford Branch; Mrs. Irene S. Yerger has been appointed Instructor at the Waterbury Branch.

Grinnell College, Grinnell, Iowa: Dr. Roscoe Woods, University of Iowa, has been appointed Visiting Professor for 1958–59; Assistant Professor W. A. Small has been promoted to Associate Professor.

University of Nebraska: Dr. Bernard Harris, Stanford University, Dr. John Kimber, Tufts University, and Mr. Hubert Hunzeker, University of Michigan, have been appointed Assistant Professors; Mr. J. M. Anderson, Iowa State College, has been appointed Instructor; Dr. D. L. Guy has been promoted to Assistant Professor.

Ohio Wesleyan University: Professor S. A. Rowland, after thirty-two years as Department Head, has retired as Professor Emeritus and will start the instruction in mathematics for Ohio State University in the new branch college at Mansfield, Ohio; Assistant Professor G. L. Thompson, Dartmouth College, has been appointed Professor.

University of South Carolina: Associate Professors Simon Green and W. A. Rutledge, University of Tulsa, and Dr. Johann Sonner, Wright-Patterson Air Force Base, Dayton, Ohio, have been appointed to associate professorships.

Mr. Eugene Albert, Union College, Schenectady, New York, has been appointed Junior Instructor at the University of Virginia.

Dr. J. F. Andrus, University of Florida, has accepted a position as Senior Mathematical Engineer with the Lockheed Aircraft Corporation, Marietta, Georgia.

Mr. P. G. Archer, University of Buffalo, has been appointed Research Assistant at Johns Hopkins University.

Professor Emil Artin, Princeton University, will be on leave of absence during 1958–59, and will be at the University of Hamburg, Hamburg, Germany.

Dr. Ward Barnes, University of North Carolina, has been appointed Instructor at the University of Massachusetts.

Mr. Martin Berman, on leave from the University of Cincinnati, is a Graduate Assistant at the University of Illinois.

Dr. M. R. Bottaccini, State University of Iowa, has been appointed Professor at the University of Arizona.

Mr. J. R. Boyd, Chance Vought Aircraft, Dallas, Texas, has been appointed Assistant Professor at Lamar State College.

Mrs. Carolina Brennan, University of California, Computer Center, Berkeley, has been appointed Instructor at the University of the Phillipines.

Dr. A. R. Brown, Jr., Air Proving Ground Center, has accepted a position as Operations Analyst with Headquarters, Air Defense Command, Colorado Springs, Colorado.

Associate Professor R. G. Buschman, University of Wichita, has been appointed Assistant Professor at Oregon State College.

Dr. John Christopher, Electro Data Corporation, Pasadena, California, has been appointed Assistant Professor at Sacramento State College.

Dr. L. W. Cohen, formerly Program Director for the Mathematical Sciences, National Science Foundation, has been appointed Professor of Mathematics and Head of the Department of Mathematics at the University of Maryland.

Professor Richard Courant has retired as scientific director of New York University's Institute of Mathematical Sciences and head of the University's mathematics department. He will become Professor Emeritus of Mathematics and Science Advisor to the University.

Dr. H. B. Curtis, Jr., Rice Institute, has been appointed Assistant Professor at the University of Texas.

Mr. C. L. Davis, General Motors Research Staff, has been appointed Associate Professor at Tri-State College, Angola, Indiana.

Dr. Martinus Esser, University of Maryland, has been appointed Assistant Professor at the University of Dayton.

Dr. J. R. Foote. Alamagordo, New Mexico, has accepted a position as Professor of Mathematics and Director of the Graduate Center, Holloman Air Force Missile Development Center, for the University of New Mexico.

Assistant Professor Ilse N. Gál, Cornell University, will be on leave at Yale University for the academic year 1958–59.

Assistant Professor I. S. Gál, on leave from Cornell University for the academic year 1958–59, has been appointed Research Associate at Yale University.

Professor José Gallego-Diaz, Escuela Especial de Ingenieros Agronomos, Madrid, Spain, has been appointed Visiting Professor at the University of Puerto Rico.

Professor R. W. Gardner, Olivet Nazarene College, has been appointed Dean of Students and Professor at the Eastern Nazarene College.

Associate Professor J. W. Hamblen, Oklahoma State University, has been employed as full-time Director of the new Computing Center at the University of Kentucky.

Mr. R. W. Hartop, University of Maine, has been awarded a Master of Science Fellowship at the Hughes Aircraft Company, Culver City, California.

Mr. J. R. Hatcher has accepted a position as Research Engineer with North American Aviation, Inc., Los Angeles, California.

Reverend F. A. Homann, University of Pennsylvania, has been appointed Instructor at Loyola College, Baltimore, Maryland.

Professor Aughtum S. Howard, Kentucky Wesleyan College, has been appointed Associate Professor at Eastern Kentucky State College.

Associate Professor L. K. Jackson, University of Nebraska, is a Visiting Associate Professor at the University of Oregon.

Miss Diane M. Johnson, University of Toronto, has been appointed Lecturer at the University of Manitoba.

Mr. W. B. Jones, Vanderbilt University, has accepted a position as Mathematician with the National Bureau of Standards, Boulder, Colorado.

Mr. E. H. Kanning III, University of Minnesota, has accepted a position as Associate Scientist with Lockheed Missile Systems Division, Sunnyvale, California.

Assistant Professor D. A. Kearns, University of Maine, has been appointed Professor and Head of the Department of Mathematics, Merrimack College.

Assistant Professor R. J. Kohlmeyer, Pratt Institute, has been appointed Associate Professor at Albright College.

Dr. George Kolettis, Jr., Northwestern University, has been appointed O.N.R. Research Associate at the University of Notre Dame.

Professor A. V. Kozak, Concord State College, has been appointed Professor of Mathematics Education, School of Education, Pennsylvania State University.

Mr. F. A. Kros, Sperry Rand, Univac Division, St. Paul, Minnesota, has been appointed Manager at Sperry Rand, Univac Division, New York.

Associate Professor Stephen Kulik, University of South Carolina, has been appointed Professor at Utah State University.

Dr. R. B. Leipnik, University of Washington, has accepted a position as Mathematician Consultant at the United States Naval Ordnance Testing Station, China Lake, California.

Assistant Professor W. D. Lindstrom, Iowa State College, has been appointed Associate Professor at Kenyon College.

Professor H. W. Linscheid, Southwestern State College, has been appointed Associate Professor at the University of Wichita.

Mr. F. J. Lorenzen, Jr., Union College, has been appointed Instructor at the University of Florida.

Mr. R. A. Lufburrow, Woods Hole Oceanographic Institute, has been appointed Assistant Professor at St. Lawrence University.

Mr. Ransom Van B. Lynch, Phillips Exeter Academy, is a Visiting Lecturer at Princeton University for one year.

Mr. B. L. McAllister, University of Wisconsin, has been appointed Assistant Professor at the South Dakota School of Mines and Technology.

Dr. G. H. Meisters, Iowa State College, has been appointed Instructor at Duke University.

Associate Professor Josephine M. Mitchell, University of Pittsburgh, has been appointed Associate Professor at Pennsylvania State University.

Mr. R. P. Mitchell, Naval Ordnance Laboratory, Corona, California, has accepted a position as Scientist with the Lockheed Aircraft Corporation, Palo Alto, California.

Mr. J. G. Moser, Purdue University, has been appointed Instructor at the Rose Polytechnic Institute.

Miss Elsie C. Muller, Morningside College, has been appointed Instructor at Iowa State College.

Mr. D. E. Myers, University of Illinois, has been appointed Associate Professor at Millikin University.

Professor Abba V. Newton, on leave from Vassar College for the academic year 1958–59, is at the University of Michigan on a National Science Foundation Science Faculty Fellowship.

Mr. R. D. Oberg, University of Minnesota, has accepted a position as Mathematician with the National Security Agency, Fort Meade, Maryland.

Mr. Enuenwemba Obi, University of Kansas City, has been appointed Assistant Professor at Bethany College.

Mr. R. K. Otnes, University of Nebraska, has accepted a position as a Computer Analyst at the Santa Monica Plant of Douglas Aircraft.

Mr. R. T. Pegis, University of Toronto, has accepted a position as Mathematician with Bausch & Lomb Optical Company, Rochester, New York. He is also an Assistant Lecturer at the University of Rochester.

Associate Professor E. J. Polak, Bucknell University, is at Princeton University on a National Science Foundation Full Faculty Fellowship.

Dr. H. J. Renggli, Tulane University of Louisiana, has been appointed Assistant Professor at Rutgers, The State University.

Assistant Professor J. D. Riley, Iowa State College, has accepted a position as Mathematician with the Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. C. D. Robbins, University of Oklahoma, has accepted a position as Research Engineer with Douglas Aircraft Company, Santa Monica, California.

Professor Henry Scheffe, University of California, will be Visiting Professor at Princeton University for the academic year 1958–59, where his work with the Statistical Techniques Research Group will be partially supported by a grant from the National Science Foundation.

Dr. E. C. Schlesinger, Yale University, has been appointed Assistant Professor at Wesleyan University.

Mr. R. L. Schwaller, Marquette University, has been appointed Instructor at the College of St. Thomas.

Mr. R. F. Shortt, Clarkson College, has been appointed Assistant Professor at the South Dakota School of Mines and Technology.

Associate Professor R. J. Silverman, Illinois Institute of Technology, is Visiting Professor at the College of Agriculture and Mechanical Arts, University of Puerto Rico.

Mr. W. E. Smith, University of California, Los Angeles, has been appointed Assistant Professor at Occidental College.

Associate Professor W. K. Smith, Antioch College, has been appointed Associate Professor at Bucknell University.

Professor S. M. Spencer, Jr., Louisiana College, has been appointed Head of the Mathematics Department at McNeese State College.

Associate Professor R. A. Struble, Illinois Institute of Technology, has been appointed Associate Professor at the State College of Agriculture and Engineering, University of North Carolina.

Mr. C. J. Struth, East Texas State Teachers College, is now Assistant Instructor at the University of Kansas.

Mr. B. K. Swartz, Massachusetts Institute of Technology, is now Research Assistant at the University of California, Los Alamos Scientific Laboratory.

Dr. W. C. Swift, Bell Telephone Laboratories, Murray Hill, New Jersey, has been appointed Assistant Professor at Rutgers, The State University.

Associate Professor Choy-tak Taam, The Catholic University of America, has been appointed Professor at Georgetown University.

Mr. Peter Terwey, Jr., on leave from Lamar State College of Technology, is at the Agricultural and Mechanical College of Texas on a National Science Foundation Science Faculty Fellowship.

Mr. R. J. Thomas, University of Illinois, has been appointed Instructor at DePauw University.

Assistant Professor Nura D. Turner, State University of New York, College for Teachers at Albany, has been promoted to Associate Professor.

Mr. R. G. Vinson, University of Tennessee, has been appointed Instructor at the University of Alabama.

Dr. Vaughan Weston, Defense Research Board, Toronto, Canada, is now Research Associate with the Radiation Laboratory, University of Michigan.

Mrs. L. M. Wheeler, Astoria High School, Astoria, Oregon, is now a mathematics teacher at Edmonds Senior High School, Edmonds, Washington.

Professor Leo Zippin, Queens College, has been appointed Professor of Mathematics at Yeshiva University's newly-created Graduate School of Mathematics.

Professor Emeritus Harry Birchenough, State University of New York, College for Teachers at Albany, died on August 20, 1958 at the age of 74. He was a charter member of the Association.

Miss Patience B. Klopp, University of Connecticut, Hartford Branch, died July 22, 1958.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The seventeenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Hofstra College, Hempstead, Long Island, April 19, 1958. Dr. J. T. Johnson, Vice-President of Hofstra College gave the address of welcome. Professor E. R. Stabler, Vice-Chairman for Colleges, chaired the morning session which was devoted to papers in applied mathematics. In the afternoon,

Dr. R. N. Walter, Vice-Chairman for High Schools, was moderator for the panel discussion on the topic "What a High School Teacher Should Know About Subjects and Techniques." There were 110 persons in attendance, including 81 members of the Association.

Professor J. N. Eastham, Chairman of the Section, presided at the business meeting. Reports were given by the Governor, Professor Jewell H. Bushey, the Treasurer, Mr. Aaron Shapiro, and the Committee on Contests and Awards. Professor E. R. Stabler resigned his office of Vice-Chairman for Colleges; no appointment was made at the meeting to fill the vacancy.

A resolution, proposed by the Treasurer, to table an amendment to the By-Laws to raise the dues of the Section to one dollar was adopted.

A resolution that a Speaker's Bureau composed of college and high school teachers be established by the Section to furnish speakers to high school mathematics clubs was proposed by the Chairman. The members of the Section approved the establishment of the Bureau on an experimental basis for one year.

It was moved and unanimously carried that a letter be addressed to the New York State Commissioner of Education (1) citing the article "New Requirements for Certification of High School Teachers" by Dr. E. K. Fretwell in the New York State Mathematics Teachers' Journal, Vol. 8, No. 2, April 1958 and (2) requesting that the two sections of The Mathematical Association of America, The Metropolitan New York Section and The Upper New York Section, be invited to present its views at all hearings concerning certification requirements of High School Mathematics Teachers in New York State.

The following papers were presented at the meeting:

1. *The nature of applied mathematics*, by Professor J. B. Keller, Institute of Mathematical Sciences, New York University, introduced by the Secretary.

Applied mathematics was defined as that science of which mathematics is a branch. This viewpoint was supported by historical evidence. Of the three main subdivisions of mathematics, it was pointed out that analysis is most important in applications. Then it was explained why an applied mathematician must know the field of application in which he works and why he must formulate the problems himself. The significance of properly-posed problems was described next and examples of such problems were presented. Once a problem is formulated and found to be properly posed, its properties must be deduced. Finally, expressions for the solution must be found.

2. *The mathematics of operations research*, by Dr. A. W. Jones, Bell Telephone Laboratories, New York City.

Papers 3, 4, and 5 were given in the panel discussion on the topic "What a High School Teacher Should Know About Subjects and Techniques."

3. *Commission on Mathematics of the College Entrance Examination Board*, by Professor M. F. Rosskopf, Teachers College, Columbia University.

The program of the Commission on Mathematics of the College Entrance Examination Board is not a radical program. However, it does imply that secondary school teachers will need to know something about mathematics that may not have been included in their preparatory program. The Commission program does use the language of sets and vectors, and does suggest that probability and statistics be included in the college preparatory program. Much emphasis is put upon clear, precise, and careful use of mathematical language. The Commission report will include not only some discussion of the way in which the program can be put into effect but also will include some suggested classroom units.

4. Panel Discussion, by Mr. Frank Hawthorne, Supervisor of Mathematics Education, The State Education Department, Albany.

Certain conditions are necessary. These include a knowledge of mathematics beyond anything he will be required to teach, an interest in and love for his subject, a knowledge of some of the related fields and a real interest in young people and their intellectual growth. Among the desirable conditions are: a knowledge of the usual sequence in analysis, certainly through the calculus; at least one college course in synthetic geometries and one in statistics. The most valuable "pedagogical" work is probably practice teaching under a skilled and experienced teacher. The existence of sufficient conditions is denied.

5. Panel Discussion, by Mrs. Roxee W. Joly, President, Mathematics Chairmen's Association of New York City, introduced by the Secretary.

As chairman of a mathematics department in a large high school, Mrs. Joly based her ideas on desirable factors of subject and technique which new teachers did not know, even though they had a regular mathematics major in college and graduate school. One neglected area was awareness of actual new syllabi, their philosophy and content. Another serious need was ability to prepare Key Lesson Plans to make the subject alive and exciting. All should take Professionalized Subject matter; a laboratory in Math Recreations, clubs, teams, assemblies; a course tying the anecdotes of history of mathematics to units in the secondary school syllabi; observation and teaching in schools where a dynamic, carefully-selected leader can help them grow in all these directions.

AZELLE B. WALTCHER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.

ILLINOIS, Millikin University, Decatur, May 8–9, 1959.

INDIANA, Valparaiso University, May 2, 1959.

IOWA, Iowa Wesleyan University, Mount Pleasant, April 17, 1959.

KANSAS

KENTUCKY, Centre College of Kentucky, Danville, April, 1959.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 13–14, 1959.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 6, 1958.

METROPOLITAN NEW YORK

MICHIGAN, Michigan State University of Agriculture and Applied Science, East Lansing, March 28, 1959.

MINNESOTA

MISSOURI, Lindenwood College, St. Charles, April 25, 1959.

NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, Stanford University, January 17, 1959.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.

PHILADELPHIA

ROCKY MOUNTAIN, Utah State University of Agriculture and Applied Science, Logan, May 8–9, 1959.

SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.

SOUTHERN CALIFORNIA, University of Redlands, March 14, 1959.

SOUTHWESTERN, Arizona State College, Tempe, Spring, 1959.

TEXAS, University of Texas, Austin, April, 1959.

UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.

WISCONSIN, Wisconsin State College, Platteville, May 2, 1959.

ACKNOWLEDGEMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

General Articles: C. B. Allendoerfer, Emil Artin, Howard Campbell, L. Cesari, R. R. Christian, R. V. Churchill, Paul Civin, N. A. Court, H. F. Davis, D. Derry, D. J. Dickinson, Trevor Evans, G. E. Forsythe, J. E. Freund, C. Froese, H. W. Gould, Leon Henkin, E. Hewitt, S. A. Jennings, M. S. Klamkin, Emma Lehmer, E. Leimanis, A. E. Livingston, N. Macon, N. S. Mendelsohn, Karl Menger, Richard Montague, L. Moser, B. N. Moyls, M. E. Munroe, S. W. Nash, R. A. Restrepo, R. A. Rosenbaum, I. J. Schoenberg, Nathan Schwid, D. E. Spencer, D. J. Struik, Olga Taussky-Todd, H. A. Thurston, L. Tornheim, R. M. Winger, R. J. Wisner, H. J. Zassenhaus.

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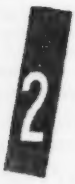

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